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AN INFINITE-SERIES REPRESENTATION FOR FUNCTIONS IN DIFFERENTIABILITY CLASS C^{∞}

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Resumen. *La representación de funciones por medio de series al infinito encuentra aplicaciones en diferentes campos de las matemáticas y de la ingeniería. La más común de estas representaciones es la serie de potencias. En este trabajo se presenta una novedosa representación de funciones continuamente diferenciables mediante series al infinito y se estudia su convergencia. Adicionalmente, presentamos algunas aplicaciones, incluyendo una forma de representar a la función gamma por medio de series al infinito.*

Palabras Claves: Series infinitas, representaciones, series de potencias, series de Taylor, funciones suaves y función gamma.

Abstract. *Infinite-series representations find applications in many mathematical and engineering domains. The most common infinite-series representation is the power series. In this paper, we present a novel infinite-series representation of smooth functions and study its convergence. Additionally, we present applications, including an infinite-series representation of the gamma function.*

Keywords: Infinite-series, representation, power series, Taylor series, smooth functions, gamma function.

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1. INTRODUCTION

Infinite-Series representation of functions find applications in many mathematical and engineering domains. For example, they are used in numerically computation of the values of functions or in estimating their behavior. Additionally, infinite-series representations are used in solving systems of differential equations.

The most common infinite-series representation of a function $f(x)$ is the power-series representation of the form

$$f(x) = \sum_{i=0}^{\infty} c_i (x-a)^i$$

for some values of c_i's, and a. It remains to find the values of x for which this representation is valid. It can be shown that if *f(x)* has a power series representation, then $c_i = \frac{f^{(i)}(a)}{i!}$, where $f^{(i)}(x)$ is the i^{th} derivative of $f(x)$. This representation is known as the Taylor series representation.

In this work, we propose a novel infinite-series representation for smooth functions and we study its convergence. It should be pointed out that the proposed series representation is, thus, not a power series. For the best of the author's knowledge, this infinite-series representation is presented here for the first time.

The rest of the paper is structured as follows. In Section II we present the main two theorems of this work. Section III provides illustrations of the proposed infinite-series representation and some of their applications. Finally, Section IV closes the paper providing the conclusions.

2. INFINITE-SERIES REPRESENTATION

A function $f(x)$ is said to be of class C^k if the derivatives $f^{(1)}(x), f^{(2)}(x), \dots, f^{(k)}(x)$ exists and are continuous. Note that the continuity of the k derivatives is automatic except for $f^{(k)}$. The function $f(x)$ is said to be of class C^{∞} or smooth it has derivatives of all orders. Furthermore, if $f(x)$ has derivative of all orders within the open interval (a,b) we say that $f(x)$ is smooth in (a,b) and write $(x) \in C^{\infty}(a,b)$.

We now present the main results of this paper.

Theorem 1: Let $f(x)$ be a function of class $C^{n+1}(a,b)$ and integrable in (a,b), then, there exists ξ in (a,b) such that

$$\int_a^b f(x)dx = \sum_{i=0}^{n} (-1)^i \frac{x^{i+1}}{(i+1)!} f^{(i)}(x) \Big|_{x=a}^{b} + (-1)^{n+1}(b-a) f^{(n+1)}(\xi) \frac{\xi^{n+1}}{(n+1)!}$$

where $f^{(0)}(x) = f(x)$.

Proof:
Define

$$F(y) = \int_a^y f(t)dt$$

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and

$$S_n(y)=\sum_{i=0}^{n}(-1)^i f^{(i)}(t)\frac{t^{i+1}}{(i+1)!}\Big|_{t=a}^{y}$$

so that we can define

$$R_n(y)=F(y)-S_n(y).$$

Differentiating $R_n(y)$ with respect to y we have

$$R_n^{(1)}(y)=F^{(1)}(y)-S_n^{(1)}(y)$$

thus

$$S_n^{(1)}(y)=\sum_{i=0}^{n}\left[(-1)^i f^{(i)}(y)\frac{y^i}{i!}+(-1)^i f^{(i+1)}(y)\frac{y^{i+1}}{(i+1)!}\right]$$

so that

$$S_n^{(1)}(y)=f(y)+(-1)^n f^{(n+1)}(y)\frac{y^{n+1}}{(n+1)!}$$

On the other hand, by the Fundamental Theorem of Calculus (see [1]) we have

$$F^{(1)}(y)=f(y).$$

Thus

$$R_n^{(1)}(y)=f(y)-\left[f(y)+(-1)^n f^{(n+1)}(y)\frac{y^{n+1}}{(n+1)!}\right]$$
$$=(-1)^{n+1} f^{(n+1)}(y)\frac{y^{n+1}}{(n+1)!}$$

and

$$R_n(y)=(-1)^{n+1}\int_a^y f^{(n+1)}(x)\frac{x^{n+1}}{(n+1)!}dx$$

By the First Mean Value Theorem for Integration (see [1]) we have

$$R_n(y)=(-1)^{n+1}(y-a)f^{(n+1)}(\xi)\frac{\xi^{n+1}}{(n+1)!}$$

for some value ξ in (x,a). The term $R_n(y)$ is called the residual of the series after $n+1$ terms. ■

Theorem 2: Let $f(x)$ be a function in class $C^{\infty}$and integrable, then

$$\int f(x)dx=\sum_{i=0}^{\infty}(-1)^i\frac{x^{i+1}}{(i+1)!}f^{(i)}(x)+C$$

Proof:

Since

$$R_n(y)=F(y)-S_n(y)$$

showing that

$$S_n(y)\to F(y) \text{ as } n\to\infty$$

is equivalent to showing

$$R_n(y)\to 0 \text{ as } n\to\infty$$

or

$$\lim_{i\to\infty}(b-a)\,f^{(n+1)}(y)\frac{y^{n+1}}{(n+1)!}=0$$

From the fact that for any y

$$e^y=\sum_{i=0}^{\infty}\frac{y^i}{i!}$$

we have

$$\lim_{n\to\infty}\frac{y^i}{i!}=0$$

On the other hand, since $f(y)$ is in class $C^{\infty}(a,b)$, then for every y in (a,b) we have that $f^{(i)}(y)<\infty$, and thus

$$\lim_{n\to\infty}(b-a)\,f^{(n+1)}(y)\frac{y^{n+1}}{(n+1)!}=0$$

which concludes the proof. ■

It should be noted that the infinite series representation introduced in *Theorem 2* can be considered as a linear combination in an infinite-dimensional functional space (see [2], [3]). We next provide some illustrations of applications of our proposed infinite-series representation.

3. ILLUSTRATIONS OF PROPOSED INFINITE-SERIES REPRESENTATION

In the first illustration we represent the exponential function with our proposed infinite-series. Then, we compute the value of a definite integral of a trigonometric function using our proposed infinite-series representation. Afterwards, the coefficients of the Fourier transform are obtained using our proposed infinite-series. Finally, a novel representation of the gamma function using our infinite-series representation is given.

A. Representing the exponential function

Consider the function$f(x)=e^x$. Based on *Theorem 2* we have

$$e^x=\sum_{i=0}^{\infty}(-1)^i\frac{x^{i+1}}{(i+1)!}e^x+C_{e^x}$$

and thus,

$$\frac{e^x-C_{e^x}}{e^x}=\sum_{i=0}^{\infty}(-1)^i\frac{x^{i+1}}{(i+1)!}$$

where it can be shown that $C_{e^x}=1$, for every x.

B. Computing a definite integral of a trigonometric function

We will now use our proposed representation to compute the value of $\int_0^{\pi/2}cos(x)dx$ using our proposed infinite-series representation.

$$\int_0^{\pi/2} cos(x)dx = \sum_{i=0}^{\infty}(-1)^i cos^{(i)}(\pi/2)\frac{(\pi/2)^{i+1}}{(i+1)!} - \sum_{i=0}^{\infty}(-1)^i cos^{(i)}(0)\frac{(0)^{i+1}}{(i+1)!}$$

$$= \sum_{i=0}^{\infty}(-1)^{2i+1}(-1)^{i+1}\frac{(\pi/2)^{(2i+1)+1}}{[(2i+1)+1]!}$$

$$= (-1)\left[\sum_{i=0}^{\infty}(-1)^i\frac{(\pi/2)^{2i}}{(2i)!} - \frac{(\pi/2)^0}{0!}\right]$$

$$= (-1)cos(\pi/2) + 1 = 1$$

C. Obtaining the coefficients of a Fourier transform

We now use our proposed infinite-series representation to obtain the coefficients of a Fourier transform.

We know that any periodic function between *-L* and *L*, integrable within this interval and with a countable number of discontinuities can be represented by an infinite-series representation of sines and cosines of the form

$$f(t) = a_0 + \sum_{i=1}^{\infty} a_i cos(i\pi t/L) + b_i sin(i\pi t/L)$$

where

$$a_0 = (2L)^{-1}\int_{-L}^{L} f(t)dt$$

$$a_i = L^{-1}\int_{-L}^{L} f(t)cos(i\pi t/L)dt$$

$$b_i = L^{-1}\int_{-L}^{L} f(t)sin(i\pi t/L)dt$$

Using *Theorem 2*, the Fourier coefficients are obtained as

$$a_i = (-1)^i(L)^{-1}\sum_{k=0}^{\infty}(-1)^k(L/i\pi)^{2k+1}f^{(2k+1)}(t)\Big|_{-L}^{L}$$

$$b_i = (-1)^i(L)^{-1}\sum_{k=0}^{\infty}(-1)^{k+1}(L/i\pi)^{2k+1}f^{(2k)}(t)\Big|_{-L}^{L}$$

D. Representing the gamma function

We now present a novel infinite-series representation of the gamma function based on our proposed representation.

Consider the gamma function $\Gamma(z)$, defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt$$

for ∈ ℝ.

We first set

$$f(t) = t^{z-1}e^{-t},$$

and observe that

$$f^{(i)}(t) = e^{-t}\left[\sum_{k=0}^{i}(-1)^k\binom{i}{k}\frac{(z-1)!}{(z-1-k)!}t^{z-1-k}\right]$$

Applying Theorem 2 we have the following gamma function representation

$$\Gamma(z) = \sum_{i=0}^{\infty}(-1)^i\frac{t^{i+1}}{(i+1)!}f^{(i)}(t)\Bigg|_{t=0}^{\infty}$$

$$= \sum_{i=0}^{\infty}(-1)^i\frac{t^{i+1}}{(i+1)!} e^{-t}\left[\sum_{k=0}^{1}(-1)^k\binom{i}{k}\frac{(z-1)!}{(z-1-k)!}t^{z-1-k}\right]\Bigg|_{t=0}^{\infty}$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\frac{(z-1)!}{(i+1)(z-1-k)!}t^{z+i-k}e^{-t}\Bigg|_{t=0}^{\infty}$$

4. CONCLUDING REMARKS

In this work, we presented a novel infinite-series representation of smooth functions and proof two related theorems. The first theorem stated that an analytical function may be represented by the proposed series, while the second theorem studied its convergence. At the outset of the paper we presented different applications of the presented results including a gamma function series representation.

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