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UNCOMMONLY BEHAVED LIPSCHITZ DOMAINS

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Resumen. *En este artículo, queremos exhibir un dominio uniformemente Lipschitziano, tal que la intersección del dominio con cualquier bola centrada en el origen (el cual es un punto de la frontera) no es un dominio Lipschitziano.*
Palabras Claves: Dominios Lipschitz , regularidad de Límites

Abstract: *This paper is about exhibiting a uniformly Lipschitz domain with the following property: The intersection of the domain with any ball centered at the origin is not a Lipschitz domain.*
Keywords: Lipschitz domains, Boundary regularity.

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1. INTRODUCCIÓN

A Lipschitz Domain is an open connected set in n–dimensional Euclidean space whose boundary is locally expressed as the graph of a Lipschitz function. Lipschitz condition on the boundary is a natural assumption of regularity, much weaker than differentiability, yet strong enough to allow the study of important properties like continuity at the boundary of solutions to Dirichlet type problems, associated with many linear and nonlinear partial differential equations. A very important result in this context is the existence and uniqueness of minimal positive harmonic functions having a “pole” at any given boundary point, proved by Hunt and Wheeden [6] (1968). The boundary behavior of harmonic functions where studied from different perspectives by Ancona [1] (1978), Benedicks [2] (1980), Jerison and Kenig [7] (1982), Wittman [8] (1985) among others.

Working with the boundary at different scales (in blow up techniques for example), we need to localize and study the behavior of solutions on small neighborhoods around a fixed boundary point. To do so, it is customary to consider intersections of the domain with small “cylindrical neighborhoods.” These kind of neighborhoods are used to preserve the Lipschitzian character of the original domain.

The downside is the need for choosing a hyperplane to express the boundary, locally, as a going to be the axis of the cylinders. Since the notations become cumbersome, we can always ask whether we can work with regular balls.

Unfortunately, the intersection of a Lipschitz domain with a ball may fail to be a Lipschitz domain, as one can easily provide examples. Nevertheless, we can still hope to work with balls, since what is needed in most problems is the existence of a sequence of balls with radius going to zero, whose intersection with the domain is a Lipschitz domain.

Even this weaker requirement is not always guarantied, but an example of a domain exhibiting such behavior is rather uncommon. Therefore, we think it is important to provide an explicit example of a domain in which the Euclidean balls produce all bad intersections and the cylindrical neighborhoods are really needed.

This paper is about exhibiting a uniformly Lipschitz domain with the following property: The intersection of the domain with any ball centered at the origin is not a Lipschitz domain. This construction is rather elementary, but we hope that the reader will find it interesting and elucidative.

This small contribution is dedicated in memory of Kai Lai Chung, a fine mathematician and mentor of many generations of mathematicians who learned Probability from his books, specially the very well known “A Course in Probability Theory” [3]. His book “Green, Brown and Probability” [4] (1995) tells the story of the ‘symbiosis’ between Analysis and Probability from Chung’s mature perspective. Starting with Green’s work of 1828, on the mathematical theory of Electricity, Chung takes the reader in a tour through random processes, random time, Markov property and he goes on discussing deeper results and showing what comes from Analysis and what is genuine Probability thought. Chung, very elucidatively, explains the

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trouble with irregular boundary points with respect to the classic Dirichlet problem and he asks this question about using spherical instead of cylindrical neighborhoods. I acknowledge him the privilege of some insightful discussions on this and other matters.

These themes are at the crossroad between Analysis and Probability. The bridge was led down by Shizuo Kakutani's solution of the Dirichlet problem by probabilistic means in 1944, which is one of the most elegant theorems showing the interplay between Probability and Analysis. A huge development came as a result and the book by J. L. Doob [5] (1983) gives a good account of it.

2. THE BASIC TWO DIMENSIONAL PROFILE

In this section, we construct a plane domain, given as the subgraph of a uniformly Lipschitz function, and find a sequence of discs, centered at the origin, with diame- ters going to zero, such that the intersection of any of those discs with the domain fails to be a Lipschitz domain.

Consider the function defined by:

(2.1) $f(r) = \alpha r\sin(\beta \ln r), \forall r > 0$

Extended by 0 at the origin. The constants α and β will be chosen later.

Note that f is uniformly Lipschitz in [0, ∞[. In fact f is Lipschitz continuous at 0 and

(2.2) $f(r) = \alpha r\sin(\beta \ln r) + \alpha r\cos(\beta \ln r), r > 0$

Is bounded.

The domain D es the subgraph

$$D = (x, y) \in \mathbb{R}^2; y < f(|x|)$$

We denote by Bδ the disc of radius δ, centered at the origin, and by Qδ the connected component of D ∩ Bδ, having the origin as a boundary point.

Let us call singular point of intersection any boundary point (x, y) ∈ ∂Qδ, where the Lipschitz condition is not satisfied.

Since the minimum of two Lipschitz functions is again Lipschitz, we can not find singular points of intersection in the upper semi-circumference. A cusp like config- uration (where the Lipschitz condition fails) may occur at a point where the graph of f touches the lower semi-circumference, and both tangents, to the graph and to the circumference, coincide. i.e.

(2.3) $y = (-\delta^2 - x^2)^{\frac{1}{2}} = f(|x|)$

And

(2.4) $\frac{x}{\delta^2 - x^2} = f'(|x|)$

There are of course other conditions to be verified by the singular points of inter- section, namely: a) The graph of *f* can not be below the lower semi-circumference

Near the point of intersection, otherwise the point is not in ∂(D ∩ B_δ). b) The cusp must be part of Q_δ and not in a separate component.

From equation 2.3, we obtain the radius of the circumference which intersects the graph of f , at (x, f |x|), for any given x. i.e.

(2.5) $\delta^2 = x^2 + f(|x|)$

By multiplying 2.3 and 2.4 side by side, we have f f 0 = −|x|, for any abscissa x of a singular point of intersection. Then the abscissas of singular points of intersection are singular points of the right hand side of 2.5. In fact, differentiating 2.5, for x = 0, we get

(2.6) $2x + 2ff' \bullet \frac{x}{|x|}$,

Which is 0 if x is the abscissa of a singular point of intersection.

Using the actual expression of f (given in 2.1), equation 2.5 becomes

(2.7) $\delta^2 = x^2 \bullet (1 + \alpha^2 \bullet \sin[2\ (\beta \ln|x|)$,

Since the right hand side of 2.7 is a function of x_2, we can simplify the calculations by considering δ_2 (let's call it g) as a function of τ = x2 . That is, we put

(2.8) $g(\tau) = \tau * (1 + \alpha^2 * \sin^2\left(\frac{\beta}{2}\ln\tau\right), \tau > 0.$

Now, we differentiate g with respect to τ to locate its singular points.

(2.9) $g(\tau) = 1 + \alpha^2 \bullet \sin^2\left(\frac{\beta}{2}\ln\tau\right) + \alpha^2\beta\sin\left(\frac{\beta}{2}\ln\tau\right) \bullet \cos\left(\frac{\beta}{2}\ln\tau\right)$

Using the well-known trigonometric identities

(2.10) $\sin(\theta) = 2 \bullet \sin\left(\frac{\theta}{2}\right) \bullet \cos\left(\frac{\theta}{2}\right)$

and $\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$,

Equation 2.9 becomes

(2.11) $g'(\tau) = 1 + \frac{\alpha^2}{2}(1 - \cos(\beta\ln\tau)) + \frac{\alpha^2}{2}\beta\sin(\beta\ln\tau)$.

Choosing

(2.12) $\beta = \tan\phi$ and $\alpha^2 = \frac{2\cos\phi}{1-\cos\phi}$,

for some fixed φ ∈]0, π [, and multiplying equation 2.11 by 1 − cos φ, we get

(2.13) $1 - \cos(\phi) \bullet g'(\tau) = 1 - \cos(\tan\phi \bullet \ln\tau) \bullet \sin\phi$

Using the identity

(2.14) $cos(\varphi + \psi) = \cos(\phi) \bullet \cos(\psi) - \sin(\phi) \bullet \sin(\psi)$

We obtain

(2.15) $1 - cos(\phi) \bullet g'(\tau) = 1 - \cos(\tan\phi\ln\tau + \phi)$.

Since 1 − cos φ > 0 and cos(θ) ≤ 1, for all θ ∈ R, equation 2.15 shows, in particular, that g, and a fortiori = δ, is a non decreasing function of τ .

All the requirements on the singular points of intersection are greatly simplified by this fact, since it means that any circumference (of any radius δ > 0), centered at the origin, intersects the graph of f exactly twice, at

$\left(\pm\sqrt{\tau}, f(\tau)\right)$

Where τ > 0 is the unique solution of 2.8 for a given δ (g = δ2).

This also implies that D ∩ Bδ is connected. So, Qδ = D ∩ Bδ , and if f (√τ) < 0,

(2.16) $Q_\delta = \left\{(x,y) \in \mathbb{R}^2; |x^2| < \sqrt{\tau}, -(\delta^2 - x^2)^{\frac{1}{2}} < y < f(|x|)\right\}$

By 2.15, the singular points of g are the solutions of

(2.17) $\cos(\tan\phi \bullet \ln\tau + \phi) = 1.$

Then, the singular set of g is implicitly defined by

(2.18) $\tan\phi \bullet \ln\tau + \phi = -2k\pi, k \in \mathbb{Z}$

(The choice of the negative sign in the left hand side is just a matter of taste.) Explicitly, 2.18 defines the sequence

(2.19) $\tau_k = \exp\left(-\frac{(\phi+2k\pi)}{\tan\phi}\right), k \in \mathbb{Z}$

which tends to 0 when k → ∞.

At those points, the function f is given by

$$f(\sqrt{\tau_k}) = \sqrt{\tau_k} \bullet \sqrt{\frac{2\cos\phi}{1-\cos\phi}} \bullet \sin\left(\frac{\phi}{2} - k\pi\right)$$

(2.20) $= (-1)^{k+1} \bullet \sqrt{\tau_k} \bullet \sqrt{\frac{2\cos\phi}{1-\cos\phi}} \bullet \sin\left(\frac{\phi}{2}\right),$

and the radius is

(2.21)

$\delta_k = \sqrt{1+\cos\phi}\exp\left(-\frac{\phi+2k\pi}{2\tan\phi}\right), k \in \mathbb{Z}$

As noted above, we obtain a cusp like configuration (two in fact) for each singular point for which f takes a negative value. Then, the singular radii are those for which k is an even number. i.e.

(2.22)

$\delta_{2k} = \sqrt{1+\cos\phi}\exp\left(-\frac{\phi+4k\pi}{2\tan\phi}\right), k \in \mathbb{Z}$

In other words, the intersection of D with any disc of radius δ2k, k ∈ Z, centered at the origin, is not a Lipschitz domain. Note that we have not chosen any particular value for φ.

3. MAIN EXAMPLE

The main example is constructed in $\mathbb{R}^3$, using the previous two dimensional profile. We think of the function f (see 2.1) as a radial function on the plane and we make it to depend on the angle as well, introducing a "phase." In polar coordinates,

(3.1)

$\tilde{f}(r,\theta) = r \bullet \sqrt{\frac{2\cos\phi}{1-\cos\theta}} \bullet \sin(\tan\theta \bullet \ln r + \theta + \phi),$

The uniform Lipschitz property is still satisfied by f̃, since

(3.2)

$\frac{\delta\tilde{f}}{\delta r}(r,\theta) = \frac{\sqrt{2}}{\sqrt{(1-\cos\theta)\cos\theta}}\sin(\tan\phi\ln r + \theta + \phi),$

And

(3.3)

$$\frac{1}{r}\frac{\delta\tilde{f}}{\delta r}(r,\theta) = \sqrt{\frac{2\cos\theta}{1-\cos\theta}}\cos(\tan\phi\ln r + \theta + \phi),$$

Are bounded.

The domain D is, in cylindrical coordinates, given by

$D = \left\{(r,\theta,z) \in \mathbb{R}^+ \times [0,2\pi] \times \mathbb{R}; z < \tilde{f}(r,\theta)\right\}$

And B_δ now denotes the ball in 3−dimensional Euclidean space, of radius δ, centered at the origin.

The phase θ introduced in the function sinus does not disturb the monotonicity (with respect to τ) of the function

$$\tilde{g}(\tau,\theta) = \tau\left(1 + \frac{2\cos\phi}{1-\cos\phi}\sin^2\left(\frac{tan}{2}\ln\tau + \theta\right)\right)$$

(3.4) $= \frac{\tau}{1-\cos\phi}(1 - \cos\phi\cos(\tan\phi\ln\tau + 2\theta))$

Since

(3.5)

$\frac{\partial\tilde{g}}{\partial\tau}(\tau,\theta) = \frac{1}{1-\cos\phi} = (1 - \cos(\tan\phi\ln\tau + 2\theta + \phi)) \geq 0.$

Then, for any δ > 0, the graph of f̃ intersects ∂B_δ , only once for every θ ∈ [0, 2π] fixed, and D ∩ B_δ is connected.

By 3.5,

(3.6) $\frac{\partial\tilde{g}}{\partial\tau}(\tau,\theta) = 0 \Leftrightarrow \tan\phi\ln\tau + 2\theta + \phi = -2k\pi, k \in \mathbb{Z}$

Then, the singular radii are given by

(3.7) $r(\theta,k) = \exp\left(-\frac{\phi+2\theta+2k\pi}{2\tan\theta}\right), k \in \mathbb{Z}, \theta \in [0,2\pi]$

Evaluating f̃ at these points, we have, for k ∈ ℤ and θ ∈ [0, 2π],

(3.8)

$\tilde{f}(r(\theta,k),\theta) = (-1)^{k+1}\exp\left(-\frac{\phi+2\theta+2k\pi}{2\tan\phi}\right)\sqrt{\frac{2\cos\phi}{1-\cos\phi}}\sin\left(\frac{\phi}{2}\right).$

For k even, 3.8 gives a negative value. i.e. f̃(r(θ, 2k), θ) < 0, for all k ∈ ℤ. Then, the points

(3.9)

$\left(r(\theta,2k),\theta,\tilde{f}(r(\theta,2k),\theta)\right), k \in \mathbb{Z}, 0 \in [0,2\pi]$

Are the apex of a cusp like region inside D ∩ $B_{\delta(\theta,k)}$, where

(3.10)

$\delta(\theta,k) = \sqrt{1+\cos\phi}\exp\left(-\frac{\phi+2\theta+4k\pi}{2\tan\phi}\right), k \in \mathbb{Z}, \theta \in [0,2\pi].$

Finally, note that θ + 2kπ gives us just about any real number as k runs in Z and θ in [0, 2π]. The same is true for

(3.11) $\ln\left(\sqrt{1+\cos\phi}\right)-\frac{\phi+2\theta+4k\pi}{2\tan\phi}$.

Then, the expression

(3.12) $\sqrt{1+\cos\phi}\,\exp\left(-\frac{\phi+2\theta+4k\pi}{2\tan\phi}\right), k\in\mathbb{Z}, \theta\in[0,2\pi]$

Produces any positive real number.

Therefore, the intersection of D with any disc centered at the origin has a boundary point where the Lipschitz condition fails.

Since the value of ϕ was left unchosen, we actually found a continuous, one parameter family of domains with the claimed property, as ϕ runs in $[0,\frac{\pi}{2}]$ and α and β vary accordingly.

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