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On the construction of Alexandroff Spaces

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Abstract Alexandroff spaces, characterized by the property that arbitrary intersections of open sets remain open, play a fundamental role in topology and its applications. This article explores different methods for constructing Alexandroff spaces, organized into several approaches. First, we begin with structural techniques including characterizations of bases, the formation of subspace, and the opposite topology. We then examine constructive operations such as intersections, products, and quotients, highlighting how the Alexandroff property is preserved. These methods provide various ways to generate new Alexandroff spaces from existing ones, shedding light on their structural properties and interactions. A subsequent section investigates constructions based on morphisms, focusing on identification and final topologies, as well as primal topologies defined via self-maps. The latter part of the work deals with order-theoretic characterizations, highlighting the correspondence between Alexandroff topologies and preorders, as well as Alexandroff topologies appearing on locally finite graphs. Throughout, emphasis is placed on the preservation of the Alexandroff condition, and the insights these perspectives offer for further applications.

Keywords Topology · Alexandroff space · Primal space

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1 Introduction

An Alexandroff space is a topological space in which the intersection of any collection of open sets is still open. This condition implies that the topology is completely determined by the neighborhood structure of individual points, making Alexandroff spaces particularly significant in order theory and discrete mathematics. Introduced by Pavel Alexandroff in 1937 [1], these spaces naturally arise in various mathematical contexts, including lattice theory, domain theory, and theoretical computer science.

Definition 1 (Alexandroff space, [2]). A topological space (X,τ) is said to be an Alexandroff space if the intersection of any collection of open sets remains an open set.

Example 1. If X is a finite set, and τ a topology on X, then (X,τ) is an Alexandroff space.

Example 2. Any set endowed with the discrete topology is Alexandroff, since every subset is open in the discrete topology.

Example 3. $\mathbb{R}$ with the usual topology is not Alexandroff. For instance, the collection $\{(-\frac{1}{n},\frac{1}{n}) : n \in \mathbb{N}\}$ is a family of open sets but its intersection is the singleton $\{0\}$, and singletons are not open in the usual topology of $\mathbb{R}$.

The first consequence of this definition is the union of any collection of closed sets is still closed. A more important consequence is that each point in an Alexandroff space has a minimal open neighborhood.

Theorem 1. *Let (X,τ) a topological space. Then (X,τ) is Alexandroff if and only if each point $x \in X$ has a minimal open neighborhood.*

Proof. $(\rightarrow)$ Let (X,τ) be an Alexandroff space and $x \in X$. Let us define the collection $\mathscr{N}_x = \cap\{O \in \tau : x \in O\}$. Then $\mathscr{N}_x$ is open for the space is Alexandroff. It is also minimal in the order given by inclusion, for any other neighborhood U of x will contain an open set G containing x, *i.e.*, $x \in G \subseteq U$, hence $\mathscr{N}_x \subseteq G \subseteq U$.

$(\leftarrow)$ Suppose that for each $x \in X$ there exists a minimal open neighborhood $\mathscr{N}_x$. Let $\{O_i \in \tau : i \in I\}$ be an arbitrary family of open sets. If $\cap_{i\in I} O_i = \varnothing$, then the intersection is open. If $x \in \cap_{i\in I} O_i$, then O_i is a neighborhood of x, hence $\mathscr{N}_x \subseteq O_i$ for all $i \in I$, that is $\mathscr{N}_x \subseteq \cap_{i\in I} O_i \subseteq O_i$. However, this is also valid for any other point in $\cap_{i\in I} O_i$, hence that every point in $\cap_{i\in I} O_i$ is interior and $\cap_{i\in I} O_i$ is open. Therefore, (X,τ) is Alexandroff. $\square$

Theorem 2. *The family of minimal open neighborhoods serves as a basis for the topology in an Alexandroff space.*

Proof. Let V be an open nonempty set in an Alexandroff space (X,τ). Hence, for every $a \in V$ there is a minimal neighborhood $\mathscr{N}_a \subseteq V$, so that $V = \bigcup \mathscr{N}_a$ for all $a \in V$. $\square$

2 Structural Constructions

This section explores structural aspects inherent to Alexandroff spaces, focusing on the rol of bases, the construction of the opposite topology, and the formation of subspaces, all examined in relation to the defining properties of Alexandroff topologies.

2.1 Basis for an Alexandroff Space

In the context of Alexandroff spaces, where arbitrary intersections of open sets are also open, the behavior of the basis becomes particularly significant. It not only determines the structure of the topology but also reflects and helps characterize the Alexandroff property.

Theorem 3 (First characterization of an Alexandroff basis). *Let X be a nonempty set. A family $\mathscr{B}$ of subsets of X is a base for an Alexandroff topology on X if and only if:*

1. $X = \bigcup_{B\in\mathscr{B}} B$;
2. *For every collection $\{B_i : i \in I\} \subseteq \mathscr{B}$ and every point $x \in \bigcap_{i\in I} B_i$, there exists a basis element $B \in \mathscr{B}$ such that $x \in B \subseteq \bigcap_{i\in I} B_i$.*

Proof. Let X be a nonempty set.

($\to$). Suppose that $\mathscr{B}$ is a basis for an Alexandroff topology on X. Property 1 is satisfied because $\mathscr{B}$ is a basis. Now consider the collection $\{B_i : i \in I\} \subseteq \mathscr{B}$ and let $x \in \bigcap_{i\in I} B_i$. Since each B_i is open, the intersection $\bigcap_{i\in I} B_i$ is open, and hence it is a union of elements of the basis, i.e.,

$$\bigcap_{i\in I} B_i = \bigcup_{j\in J} B_j$$

with $B_j \in \mathscr{B}$, and therefore there exists j_0 such that $x \in B_{j_0}$.

($\leftarrow$). Suppose that $\mathscr{B}$ satisfies properties 1 and 2. Then if $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$ we have $x \in B_3 \subseteq B_1 \cap B_2$ where $B_3 \in \mathscr{B}$ by 2. Therefore, $\mathscr{B}$ is a base for a topology $\tau(\mathscr{B})$ on X.

Moreover, let $\{O_i : i \in I\} \subseteq \tau$. If $\cap_{i\in I} O_i = \varnothing$ then the intersection is open. Otherwise, let $x \in \cap_{i\in I} O_i$. Then, for each $i \in I$ there exists $B_i \in \mathscr{B}$ such that $x \in B_i \subseteq O_i$. Then there is $B \subseteq \cap_{i\in I} B_i \in \mathscr{B}$ and $x \in B \subseteq \cap_{i\in I} O_i$. Since x is arbitrary, we deduce that $\cap_{i\in I} O_i$ is also open, hence $\tau(\mathscr{B})$ is Alexandroff. $\square$

In fact, the second condition of the previous theorem is equivalent to requiring that for each $x \in X$, the intersection

$$\bigcap \{B \in \mathscr{B} : x \in B\}$$

belongs to $\mathscr{B}$, which leads to the following characterization.

Theorem 4 (Second characterization of an Alexandroff basis). *Let X be a nonempty set. A family $\mathscr{B}$ of subsets of X is a base for an Alexandroff topology on X if and only if:*

1. $X = \bigcup_{B \in \mathscr{B}} B$;
2. For every $x \in X$, $\bigcap \{B \in \mathscr{B} : x \in B\} \in \mathscr{B}$.

Proof. We only need to prove the equivalence of the second condition. If it holds that for every collection $\{B_i : i \in I\} \subseteq \mathscr{B}$ and every point $x \in \bigcap_{i \in I} B_i$, there exists a basis element $B \in \mathscr{B}$ such that $x \in B \subseteq \bigcap_{i \in I} B_i$, then for each x there exists $B_x \in \mathscr{B}$ such that

$$B_x \subseteq \bigcap \{B \in \mathscr{B} : x \in B\}.$$

But since

$$\bigcap \{B \in \mathscr{B} : x \in B\} \subseteq B_x,$$

it follows that

$$B_x = \bigcap \{B \in \mathscr{B} : x \in B\}.$$

Conversely, if for every $x \in X$ the intersection $\bigcap \{B \in \mathscr{B} : x \in B\}$ belongs to $\mathscr{B}$, then for any collection $\{B_i : i \in I\} \subseteq \mathscr{B}$ and any $x \in \bigcap_{i \in I} B_i$, we choose

$$B_x = \bigcap \{B \in \mathscr{B} : x \in B\},$$

and clearly $x \in B_x \subseteq \bigcap_{i \in I} B_i$

□

As a result, any family $\mathscr{B}$ that meets the conditions outlined above generates an Alexandroff topology, as it guarantees that each point possesses a smallest open neighborhood. We now present an example illustrating the construction of basis for an Alexandroff topology

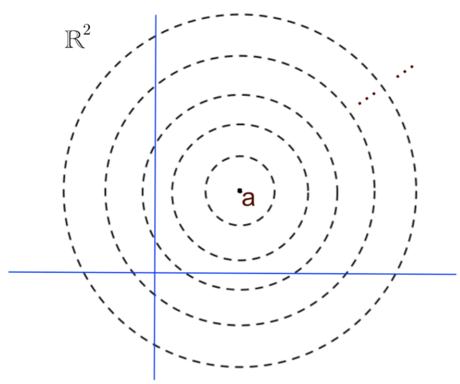

Fig. 1 Example 4: A numerable family of concentric *balls* can be a basis for an Alexandroff topology on $\mathbb{R}^2$.

Example 4. Let $X = \mathbb{R}^2$, and consider the Euclidean distance function $d : X \times X \to \mathbb{R}$. For a fixed point $a \in X$, we define the set

$$B(a,n) = \{x \in \mathbb{R}^2 : d(a,x) \leq n; n \in \mathbb{N}\}$$

as a "ball" centered at a with radius n. The collection $\mathscr{B} = \{B(a,n)\}_{n\in\mathbb{N}}$ then forms a basis for a topology on $\mathbb{R}^2$. This topology is Alexandroff, since every point $x \in \mathbb{R}^2$ possesses a minimal open neighborhood within the topology generated by $\mathscr{B}$.

2.2 The Opposite Topology

Alexandroff spaces naturally come in pairs. Because open sets remain open under arbitrary intersections, their complementary closed sets must also be closed under arbitrary unions. This property gives rise to what is known as the opposite topology.

Theorem 5. *If (X, τ) be an Alexandroff space, then the collection τ_o defined as:*

$$\tau_o = \{X \setminus U : U \in \tau\}$$

is also an Alexandroff topology on X.

Proof. It is readily seen that X and $\varnothing$ belong to τ_o, as $\varnothing, X \in \tau$.

- Let $V_1, V_2 \in \tau_o$, thus $V_i = X \setminus U_i$ with $U_i \in \tau$. Then $V_1 \cap V_2 = X \setminus U_1 \cap X \setminus U_2 = X \setminus (U_1 \cup U_2)$. Hence $V_1 \cap V_2 \in \tau_o$, for $U_1 \cup U_2 \in \tau$.
- Suppose $\{V_i\}_{i\in I}$ is an arbitrary collection of open sets in τ_o, so that $V_i = X \setminus U_i$ with $U_i \in \tau$ for all $i \in I$. Then:

$$\bigcup_{i\in I} V_i = \bigcup_{i\in I} X \setminus U_i = X \setminus \bigcap_{i\in I} U_i$$

  and $\bigcap_{i\in I} U_i \in \tau$ for τ is an Alexandroff topology, hence the union belong to τ_o.

Therefore, τ_o is hitherto a topology on X. However, the intersection $\bigcap_{i\in I} V_i$ of an arbitrary family of open sets $\{V_i\}_{i\in I}$ in τ_o is also open, for:

$$\bigcap_{i\in I} V_i = \bigcap_{i\in I} X \setminus U_i = X \setminus \bigcup_{i\in I} U_i$$

and $\bigcup_{i\in I} U_i$ is closed in τ. Consequently, (X, τ_o) is an Alexandroff space. □

2.3 Subspaces

For a topological space (X, τ) and a subset $Y \subset X$, a topology can be induced on Y by restricting the open sets of X to their intersections with Y. That is, for each open set O in X, the set $A = O \cap Y$ forms an open set in Y. The collection of such subsets, denoted τ_Y, defines a topology on Y, known as the relative topology. In this context, Y is referred to as a subspace of X.

Theorem 6. *Let (X,τ) be an Alexandroff space and $Y \neq \varnothing$ a subset of X. Then τ_Y is Alexandroff in Y.*

Proof. Since $Y \neq \varnothing$, take $y \in Y \subset X$, and let $\{\mathscr{U} \in \tau : y \in \mathscr{U}\}$ be the collection of all open neighborhoods containing y. Then, each $\mathscr{V} = \mathscr{U} \cap Y$ is an open neighborhood of y in the relative topology τ_Y. Defining $\mathscr{N}_y$ as the intersection:

$$\mathscr{N}_y = \bigcap \{\mathscr{V} \in \tau_Y : y \in \mathscr{V}\} = \bigcap \{\mathscr{U} \cap Y \in \tau_Y : y \in \mathscr{U}\}$$

it follows that $\mathscr{N}_y$ is minimal with respect to the inclusion order, $\mathscr{N}_y \neq \varnothing$ for it contains y, and it is open in the relative topology. Hence, $\mathscr{N}_y$ is a minimal open neighborhood of y. The same reasoning applies to every $y \in Y$, therefore τ_Y is Alexandroff in Y by virtue of Theorem 1. $\square$

3 Constructive Operations in Alexandroff Spaces

This section focuses on constructions via operations involving Alexandroff spaces, that is, intersections, products, quotients.

3.1 Intersections

For a nonempty set X equipped with two topologies, τ_1 and τ_2, it is a well-established result that their intersection $\tau_1 \cap \tau_2$ also forms a topology on X. This property extends to arbitrary intersections of topologies on X as well. Let $\{\tau_i : i \in I\}$ be an arbitrary collection of topologies over X, and let $\tau = \cap_{i\in I}\tau_i$. It can be shown that X and $\varnothing$ belong to τ since they belong to each τ_i in the collection. If $\{U_i\}_i$ is an arbitrary family of open sets in τ, then each U_i belongs to each one of the topologies τ_i. Therefore $\cup_{i\in I}U_i$ is open in τ_i for all $i \in I$, hence it is open in τ. Last but not least, if U_1 and U_2 are open sets in τ, they are also open sets on each τ_i in the collection; therefore $U_1 \cap U_2 \in \tau$.

Theorem 7 (Intersection of Alexandroff spaces). *Any intersection of Alexandroff topologies over X is also an Alexandroff topology on X.*

Proof. Let $\{\tau_i\}_{i\in I}$ be a collection of Alexandroff topologies over X. It is well established that $\tau = \bigcap \tau_i$ is a topology over X as well.

Let $x \in X$ and $\{U_j : x \in U_j \in \tau\}$ the collection of all open neighborhoods containing x. Then $\mathscr{N}_x = \bigcap \{U_j : x \in U_j\}$ is open and a neighborhood of x, being also minimal in the sense of the inclusion order. If V is any other open neighborhood of

x in τ, then $V \in \tau_i$ for all $i \in I$. This means that $V \in \{U_j : x \in U_j\}$ and $\mathscr{N}_x \subseteq V$. Therefore, (X, τ) is Alexandroff. □

Corolary. Finite intersection of Alexandroff topologies is also Alexandroff.

3.2 Products

Consider the n topological spaces $(X_1, \tau_1), (X_2, \tau_2), ..., (X_n, \tau_n)$, and define $X = \prod_{i=1}^{n} X_i$ as the cartesian product $X_1 \times X_2 \times ... X_n$. It has been established that a topology on X can be obtained based on the topologies of each factor X_i in the product. In the case of finite products, it has been proven that the *Tychonoff* topology and the box topology coincide, as both are generated by a common basis of the form:

$$\mathscr{B} = \{O = O_1 \times O_2 \times ... O_n : O_i \text{ is open in } X_i\}$$

Let $x = (x_1, x_2, ... x_n) \in X$ and V is a neighborhood of x, then there exists a set of the form $V_1 \times V_2 \times ... \times V_n$ subset of V, such that V_i is a neighborhood of $x_i \in X_i$. Hence there exist open sets O_i with $x_i \in O_i \subseteq V_i$ for all $i = 1...n$.

Theorem 8 (Finite product of Alexandroff spaces). *If $(X_1, \tau_1), (X_2, \tau_2), ..., (X_n, \tau_n)$ are n Alexandroff spaces, then the product $X = \prod_{i=1}^{n} X_i$ is also an Alexandroff space.*

Proof. Consider an element $x = (x_1, x_2, \dots, x_n)$ in the Cartesian product $X = \prod_{i=1}^{n} X_i$. Since each space (X_i, τ_i) is an Alexandroff space, every x_i has a minimal open neighborhood, denoted by $\mathscr{N}_{x_i} \subseteq X_i$, which forms a neighborhood basis at x_i. Given that the product of neighborhood bases at each x_i results in a neighborhood basis at $(x_1, x_2, ... x_n)$, it follows that the collection $\{\mathscr{N}_{x_1} \times \mathscr{N}_{x_2} \dots \times \mathscr{N}_{x_n}\}$ constitutes a neighborhood basis at x. Moreover, since each of these neighborhoods is minimal, the entire space X also satisfies the Alexandroff property. □

In the product space, one can define the **canonical projection** $\pi_i : X \to X_i$ for each coordinate i. Given an element $a = (a_1, a_2, \dots, a_n) \in X$, the projection is defined as $\pi_i(a) = a_i$. Now, if U_i is an open set in τ_i containing a_i, then its preimage under π_i is given by:

$$\pi_i^{-1}(U_i) = X_1 \times X_2 \times \dots \times X_{i-1} \times U_i \times X_{i+1} \times \dots \times X_n.$$

Since $\pi_i^{-1}(U_i)$ is open in X, it follows that the projection maps are continuous. Consequently, any open set in X can be expressed as:

$$O = O_1 \times O_2 \times \dots \times O_n = \pi_1^{-1}(O_1) \cap \pi_2^{-1}(O_2) \cap \dots \cap \pi_n^{-1}(O_n).$$

This naturally leads to Tychonoff's theorem, which generalizes this result to arbitrary products of topological spaces.

Definition 2 (Tychonoff's product topology). Let $\{(X_i, \tau_i)\}_{i\in I}$ be an arbitrary collection of topological spaces. The product space $X = \prod_{i\in I} X_i$ is endowed with a topology whose basis consists of open sets of the form:

$$\bigcap_{k=1}^{K} \pi_{i_k}^{-1}(O_{i_k})$$

where $K \in \mathbb{N}$ and each O_{i_k} is an open set in τ_{i_k}.

The following counterexample demonstrates that an arbitrary product of Alexandroff spaces is not necessarily Alexandroff.

Counterexample [6]. Let $W = \{x, y\}$ and (W, τ_W) be a Sierpiński space, namely, $\tau_W = \{\varnothing, \{x\}, W\}$. Then (W, τ_W) is an Alexandroff space as well. Consider the following product:

$$\prod_{n\in\mathbb{N}} W_n$$

with $W_n = W$. In Tychonoff's product topology, only a finite number of components in a product of open sets can differ from the entire space. This allows for the construction of the following family of open sets $\{O_n\}_{n\in\mathbb{N}}$:

$$\begin{aligned}
O_1 \quad &= \{x\} \times W \times W \times \ldots \\
O_2 \quad &= W \times \{x\} \times W \times \ldots \\
O_3 \quad &= W \times W \times \{x\} \times \ldots \\
&\vdots \\
O_n &= W \times \ldots \times \{x\} \ldots \times W \times \ldots
\end{aligned}$$

In each O_n, the only open set differing from W is $\{x\}$ in the n^{th} position, for every $n \in \mathbb{N}$. Consequently, each O_n is open in Tychonoff's product topology, and $\{O_n\}_{n\in\mathbb{N}}$ forms an arbitrary family of open sets. However, the intersection

$$\bigcap_{n\in\mathbb{N}} O_n = \{x\} \times \{x\} \times \{x\} \times \ldots$$

is not an open set, as it does not satisfies the definition 2. Consequently, the given product space is not Alexandroff.

3.3 Quotients

Given a topological space (X, τ) and an equivalence relation $\sim$ on X, the quotient set, denoted by $X/_\sim$, consists of all equivalence classes under $\sim$. For any $a \in X$, the notation $[a]$ represents the equivalence class of a, meaning

$$[a] = \{x \in X \mid x \sim a\}.$$

The equivalence relation naturally defines a surjective function known as the **canonical quotient map**, given by $\mathsf{q} : X \to X/_{\sim}$, where $\mathsf{q}(a) = [a]$. Similarly, the **quotient topology** in $X/_{\sim}$ is defined so that a set $U \subseteq X/_{\sim}$ is open if and only if its preimage under q, $\mathsf{q}^{-1}(U)$, is open in X.

Define the collection $\tau_{\mathsf{q}} = \{O \subseteq X/_{\sim} : \mathsf{q}^{-1}(O) \ \textit{is open in } X\}$. Then $\varnothing$ and $X/_{\sim}$ are in τ_{q}, for $\mathsf{q}^{-1}(\varnothing) = \varnothing$ and $\mathsf{q}^{-1}(X/_{\sim}) = X$, both open in X. Let $\{O_i\}_{i\in I}$ be an arbitrary collection of elements of τ_{q}. Since $\mathsf{q}^{-1}(O_i)$ is open in X for each $i \in I$, then $\bigcup_i O_i$ is in τ_{q}, because $\mathsf{q}^{-1}(\bigcup_i O_i) = \bigcup_i \mathsf{q}^{-1}(O_i)$ is open in (X,τ). Finally, if $O_1, O_2 \in \tau_{\mathsf{q}}$, then $\mathsf{q}^{-1}(O_1)$ and $\mathsf{q}^{-1}(O_2)$ are open in X; hence $O_1 \cap O_2 \in \tau_{\mathsf{q}}$, as $\mathsf{q}^{-1}(O_1 \cap O_2) = \mathsf{q}^{-1}(O_1) \cap \mathsf{q}^{-1}(O_2)$ is open in X. Consequently, τ_{q} defines a topology on $X/_{\sim}$, known as the quotient topology.

Theorem 9. *Let (X,τ) be an Alexandroff space and $\sim$ an equivalence relation on X. Then, with the quotient topology τ_{q}, the space $X/_{\sim}$ remains an Alexandroff space.*

Proof. Let $\{O_i\}_{i\in I}$ be an arbitrary family of open sets in τ_{q}. By definition of quotient topology, the preimage $\mathsf{q}^{-1}(O_i)$ is open in X for each $i \in I$. Since X is an Alexandroff space, it follows that the intersection $\bigcap_i \mathsf{q}^{-1}(O_i)$ remains open in X. Consequently, $\mathsf{q}^{-1}(\bigcap_i O_i) = \bigcap_i \mathsf{q}^{-1}(O_i)$ is open, which implies that $\bigcap_i O_i$ is open in τ_{q}. Thus, the quotient space $(X/_{\sim}, \tau_{\mathsf{q}})$ satisfies the Alexandroff property. □

4 Construction based on morphisms

This section looks into how topologies can be induced or transferred through functions, such as identification maps and families of continuous functions. Specifically, the identification topology arises from a surjective function, while the final topology is induced by a family of maps making them jointly continuous. Lastly, the primal topology is derived via a self-map $f : X \to X$. These approaches emphasize the role of morphisms in shaping the topology of a space, rather than just manipulating open sets directly.

4.1 Identification Topologies

Let X and Y be topological spaces, and let $p : X \to Y$ be a continuous function. The map p is said to be an *identification* if, for every subset $V \subseteq Y$, the openness of $p^{-1}(V)$ in X ensures that V is open in Y. As noted by Mendelson in [3], the concept of identification provides a fundamental approach to defining a topology on a set using surjective functions.

Definition 3 (Identification topology, [3]). Let (X,τ_X) be a topological space, and let $p : X \to Y$ be a surjective function onto a set Y. The *identification topology* on Y

is defined as the collection of subsets $A \subseteq Y$ for which the preimage $p^{-1}(A)$ is open in X.

Thus, p gets to be an identification map from X to Y. Let τ_Y be the set of all subsets $A \subseteq Y$ for which $p^{-1}(A)$ is open in X. It follows that $Y \in \tau_Y$ since $p^{-1}(Y) = X$ is open in τ_X, and similarly, $\varnothing \in \tau_Y$ as well. Given an arbitrary collection $\{A_i\}_{i\in I}$ of sets in τ_Y, each preimage $p^{-1}(A_i)$ is open in X, implying that $\bigcup_i A_i$ also belongs to τ_Y, as

$$p^{-1}(\bigcup_i A_i) = \bigcup_i p^{-1}(A_i)$$

remains open in X. Furthermore, for any two sets $A_1, A_2 \in \tau_Y$, their intersection $A_1 \cap A_2$ is also in τ_Y, since

$$p^{-1}(A_1 \cap A_2) = p^{-1}(A_1) \cap p^{-1}(A_2)$$

is open in X. Consequently, τ_Y forms a topology on Y, known as the *identification topology*.

Several identification examples have already been presented, such as the quotient map q from X to $X/_\sim$ and the canonical projection π_k from $\prod_i X_i$ to X_k.

Theorem 10 ([6]). *If (X, τ_X) is an Alexandroff space and $p : X \to Y$ is a surjective map that defines the identification topology τ_Y on Y, then (Y, τ_Y) inherits the Alexandroff property from (X, τ_X).*

Proof. Consider an arbitrary family of open sets $\{A_i\}_{i\in I}$ in Y, then $\bigcap_{i\in I} A_i$ is open in τ_Y, since

$$p^{-1}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} p^{-1}(A_i)$$

is open in X on account of X being an Alexandroff space. $\square$

4.2 Final Topology

The concept of identification extends to a family of surjective maps from a collection of topological spaces onto a common set, leading to what is known as the *final topology*. Given a set Y and a family of topological spaces (X_i, τ_{X_i}) with associated functions $\mathscr{F} = \{f_i : X_i \to Y\}$, the final topology $\tau_{\mathscr{F}}$ on Y is the finest topology that ensures that each f_i remains continuous. Explicitly, a set U belongs to $\tau_{\mathscr{F}}$ if and only if $f_i^{-1}(U)$ is open in X_i for all $i \in I$. Consequently, the final topology can be viewed as an indexed intersection of identification topologies. Therefore, if the spaces (X_i, τ_{X_i}) are Alexandroff, then $(Y, \tau_{\mathscr{F}})$ also inherits the Alexandroff property.

4.3 Primal Topologies

A notable category of Alexandroff spaces is known as *primal spaces*, initially referred to as *functional Alexandroff spaces* in a 2011 paper by Shirazi and Golestani [4]. The term *primal space* was later introduced independently by Echi in 2012 [5].

Given a non-empty set X and a function $f : X \to X$ we say that τ_f is the primal topology generated by f if the open sets in τ_f is the family $\{A \subseteq X : f^{-1}(A) \subset A\}$.

Proposition 1. *τ_f is an Alexandroff topology.*

Proof. 1. $\emptyset \in \tau_f$ since $f^{-1}(\emptyset) = \emptyset$.
2. $X \in \tau_f$ because $f^{-1}(X) = X$.
3. Consider the arbitrary family of open sets $\{A_i\}_{i \in I}$, then

$$f^{-1}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{-1}(A_i) \subseteq \bigcup_{i\in I} A_i,$$

the union of arbitrary family of open sets is an open set.
4. Consider the arbitrary family of open sets $\{A_i\}_{i \in I}$, then

$$f^{-1}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f^{-1}(A_i) \subseteq \bigcap_{i\in I} A_i,$$

the intersection of arbitrary family of open sets is an open set.

□

Example 5. Consider $X = \mathbb{N}$ and the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} 3n+1 \ ; \ if\ n\ is\ odd \\ \ \ n/2 \ \ \ ; \ if\ n\ is\ even. \end{cases}$$

This function induces an Alexandroff topological space $(\mathbb{N}, \tau_f)$ with a primal topology. This Alexandroff space was studied in [8], where the famous Collatz conjecture is characterized in terms of compactness and connectedness.

5 Alexandroff topologies via pre order

Given a set equipped with a preorder $(X, \leq)$, for an element x we define the set of elements greater than or equal to it as:

$$\uparrow x = \{y \in X : x \leq y\}.$$

Let the collection $B = \{\uparrow x : x \in X\}$. We show that this is a basis for a topology τ on X [7].

Proposition 2. *Let $(X, \leq)$ be a preorder, the collection $B = \{\uparrow x : x \in X\}$ is a basis for an Alexandroff topology.*

Proof. We verify the two necessary conditions for B to be a basis:

(1) Covers the space: Let $x \in X$. Since the preorder $\leq$ is reflexive, we have $x \leq x$, hence $x \in \uparrow x$. This implies that every point $x \in X$ belongs to some set in the collection B, that is,

$$X = \bigcup_{x \in X} \uparrow x.$$

(2) Local intersections: Let $\uparrow a, \uparrow b \in B$ such that $x \in \uparrow a \cap \uparrow b$. Then $a \leq x$ and $b \leq x$.

Let $c := x$, so $x \in \uparrow c \in B$. We want to show that:

$$\uparrow c \subseteq \uparrow a \cap \uparrow b.$$

Indeed, if $y \in \uparrow c$, then $c \leq y$, i.e., $x \leq y$. Since $a \leq x \leq y$ and $b \leq x \leq y$, by transitivity of the preorder it follows that $a \leq y$ and $b \leq y$, hence $y \in \uparrow a \cap \uparrow b$. Thus,

$$\uparrow x \subseteq \uparrow a \cap \uparrow b,$$

and since $x \in \uparrow x$, the condition is satisfied.

Therefore, B is a basis for a topology on X.

Now we will show that it is an Alexandroff topology:

We claim that arbitrary intersections of open sets $U = \bigcap_{i \in I} U_i$ are still open. Suppose $x \in U$, then $x \in \uparrow x \subseteq U_i$ for each i. Hence,

$$U = \bigcup_{x \in U} \uparrow x,$$

so U is open. □

6 Alexandroff topologies via graphs

Consider a locally finite graph G with no isolated vertices. For each vertex $x \in V(G)$, we denote by A_x the set of vertices adjacent to x. We construct the collection

$$S_G = \{A_x : x \in V(G)\}.$$

Since G has no isolated vertices, we have $V(G) = \bigcup_{x \in V(G)} A_x$, and therefore the collection S_G is a subbasis for a topology. We will prove that this topology is Alexandroff.

Theorem 11 ([9]). *Given a locally finite graph G, the topology generated by the subbasis S_G is an Alexandroff topology.*

Proof. It is enough to prove that the arbitrary intersection of elements in S_G is open. Let T be a subset of vertices of G. If $x \in \bigcap_{t \in T} A_t$, then $x \in A_t$ for each $t \in T$, which implies that each $t \in A_x$, and hence $T \subseteq A_x$. Since G is locally finite, A_x is finite, and thus T is also finite. This means that if T is infinite, the intersection is empty; but if T is finite, then $\bigcap_{t\in T} A_t$ is a finite intersection of open sets, and therefore open. $\square$

Since this is an Alexandroff topology on the set of vertices, a natural question is: what is the minimal neighborhood of a vertex x?

Proposition 3. *Let G be a graph. For each vertex x of G, the set $U_x = \bigcap_{y\in A_x} A_y$ is the minimal neighborhood of x.*

Proof. Since S_G is a subbasis, the minimal neighborhood of x is $U_x = \bigcap_{y\in T} A_y$ for some set T of vertices. This implies that $x \in A_y$ for each $y \in T$, and hence $T \subseteq A_x$. Therefore, $\bigcap_{y\in A_x} A_y \subseteq U_x$. $\square$

Example 6 (Alexandroff topology generated by a finite graph). Let G be a graph with vertex set

$$V(G) = \{a,b,c,d,e\}$$

and edge set

$$E(G) = \big\{\{a,b\},\{a,c\},\{b,d\},\{c,d\},\{d,e\}\big\}.$$

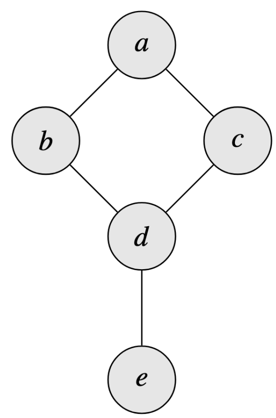

Fig. 2 Example 6: Graph induces Alexandroff topology.

This graph is finite, locally finite (each vertex has finite degree), and has no isolated vertices.

For each vertex $x \in V(G)$, we define the adjacency set

$$A_x = \{y \in V(G) : \{x,y\} \in E(G)\}.$$

The adjacency sets are:

$$A_a = \{b,c\},\ A_b = \{a,d\},\ A_c = \{a,d\},\ A_d = \{b,c,e\},\ A_e = \{d\}.$$

The collection $S_G = \{A_x : x \in V(G)\}$ is a subbasis for a topology on $V(G)$. This topology is an Alexandroff topology, since it is generated by arbitrary (finite, in this case) intersections of sets in the subbasis.

The minimal neighborhoods $U_x = \bigcap_{y \in A_x} A_y$ for each vertex are:

$$\begin{aligned}
U_a &= A_b \cap A_c = \{a, d\}, \\
U_b &= A_a \cap A_d = \{b, c\}, \\
U_c &= A_a \cap A_d = \{b, c\}, \\
U_d &= A_b \cap A_c \cap A_e = \{d\}, \\
U_e &= A_d = \{b, c, e\}.
\end{aligned}$$

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