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A Strongly Connected Topology on the Mersenne Numbers

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Abstract In the present work, we introduce a new topology on the set of Mersenne numbers. This topology is Strongly Connected in the sense that it is hyperconnected and ultraconnected. Furthermore, we characterize the conjecture of the infinitude of Mersenne primes.

Abstract En el presente trabajo introducimos una nueva topología sobre el conjunto de los números de Mersenne. Esta topología es Strongly Connected en el sentido de que es hiperconexa y ultraconexa. Más aún, con esta topología, caracterizamos la conjetura de la infinitud de los primos de Mersenne.

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1 Introduction and Preliminaries

We denote by $\mathbf{N}$ the set of natural numbers, by $\mathbf{P}$ the set of prime numbers, and by $\mathbf{M}$ the set of Mersenne numbers, defined as:

$$\mathbf{M} := \{2^n - 1 : n \in \mathbf{N}\}.$$

The set of Mersenne numbers has been widely studied by mathematicians around the world for many years, particularly due to the significant interest in Mersenne primes, denote by $\mathbf{MP} := \mathbf{M} \cap \mathbf{P}$, and their connection to perfect numbers. For a comprehensive overview of Mersenne numbers, their properties, open problems, and more, see [1].

In [4], a new topology τ is introduced on the set of positive integers generated by the base

$$\beta := \{\sigma_n : n \in \mathbf{N}\}, \text{ where } \sigma_n := \{m \in \mathbf{N} : \gcd(n,m) = 1\}.$$

The topology τ and the results presented in this section will be essential for the study of the topology we will introduce on $\mathbf{M}$.

Let $\mathbf{X} := (\mathbf{N}, \tau)$. From [4], we have the following properties about the topological space $\mathbf{X}$.

Theorem 1. *The following propositions hold:*

1. *The topological space* $\mathbf{X}$ *is hyperconnected, ultraconnected, and does not satisfy the* T_0 *separation axiom. Moreover, the tuple* $(\mathbf{X}, \cdot)$ *is a topological monoid, where* $\cdot$ *is the usual product on* $\mathbf{N}$.
2. *For every positive integer n,* $\mathbf{Cl}_{\mathbf{X}}(\{n\}) = \bigcap_{\substack{p \in \mathbf{P} \\ p \mid n}} p\mathbf{N}$*, where* $\mathbf{cl}_{\mathbf{X}}(\{n\})$ *is the closure in* $\mathbf{X}$ *of the singleton set* $\{n\}$ *and* $p\mathbf{N} := \{pn : n \in \mathbf{N}\}$.

Remark 1. Hyperconnectedness refers to a topological space with the property that there are not two non-trivial disjoint open sets in the space, and ultraconnectedness means that there are no two non-trivial disjoint closed sets in the space.

In [5], a new topological proof of the infinitude of prime numbers is presented, different from the proofs given by Fürstenberg [2] and Golomb [3], using the topology τ. Furthermore, the following characterization is presented.

Theorem 2. *[5, Theorem 4] Let* $A \subset \mathbf{P}$ *be non-empty. Then, A is dense in* $\mathbf{X}$ *if and only if A is infinite.*

The aim of this brief manuscript is to introduce a strongly connected topology $\tau_{\mathbf{M}}$ on the Mersenne numbers, study its properties, and characterize the conjecture of the infinitude of Mersenne primes in said topology.

2 The topology $\tau_{\mathbf{M}}$: Properties

First, let's define the topology $\tau_{\mathbf{M}}$. For every positive integer n, consider the set

$$\mathscr{O}_n := \{2^m - 1 \in \mathbf{M} : \gcd(2^n - 1, 2^m - 1) = 1\}.$$

Let $\beta_{\mathbf{M}} := \{\mathscr{O}_n : n \in \mathbf{N}\}$.

Theorem 3. *$\beta_{\mathbf{M}}$ is a base for some topology on* $\mathbf{M}$.

Proof. It is well known that for every positive integer n and m, $\gcd(2^n - 1, 2^m - 1) = 1$ if and only if $\gcd(n,m) = 1$, see [1, p. 114]. From this, it is easy to verify that:

1. $\bigcup_{n\in\mathbf{N}} \mathscr{O}_n = \mathscr{O}_1 = \mathbf{M}$.
2. For every $\mathscr{O}_n, \mathscr{O}_m \in \beta_{\mathbf{M}}$, it holds that $\mathscr{O}_{nm} = \mathscr{O}_n \cap \mathscr{O}_m$, since for every positive integer n,m,x, $\gcd(x,n) = \gcd(x,m) = 1$ if and only if $\gcd(x,nm) = 1$.

Therefore, $\beta_{\mathbf{M}}$ is a base for some topology on $\mathbf{M}$.

Let $\tau_{\mathbf{M}}$ be the topology generated by $\beta_{\mathbf{M}}$.

The following theorem provides us with a lot of information about the properties of $\tau_{\mathbf{M}}$.

Theorem 4. *Let $\mathbf{Y} := (\mathbf{M}, \tau_{\mathbf{M}})$. Then $\mathbf{Y}$ is homeomorphic to* $\mathbf{X}$

Proof. Consider the function $f : \mathbf{N} \to \mathbf{M}$ defined by $f(n) = 2^n - 1$. We claim that f is a homeomorphism between $\mathbf{X}$ and $\mathbf{Y}$.

The fact that f is a bijection between $\mathbf{N}$ and $\mathbf{M}$, and that its inverse function $g : \mathbf{M} \to \mathbf{N}$ is given by $g(2^n - 1) = n$, is clear.

Moreover, the continuity of f follows from the fact that for any $\mathscr{O}_n \in \beta_{\mathbf{M}}$, we have:

$$\begin{aligned} f^{-1}(\mathscr{O}_n) &= \{m \in \mathbf{N} : 2^m - 1 \in \mathscr{O}_n\} \\ &= \{m \in \mathbf{N} : \gcd(2^m - 1, 2^n - 1) = 1\} \\ &= \{m \in \mathbf{N} : \gcd(m,n) = 1\} = \sigma_n. \end{aligned}$$

On the other hand, the continuity of g is proved similarly to f. Indeed,

$$\begin{aligned} g^{-1}(\sigma_n) &= \{2^m - 1 \in \mathbf{M} : m \in \sigma_n\} \\ &= \{2^m - 1 \in \mathbf{N} : \gcd(m,n) = 1\} \\ &= \{2^m - 1 \in \mathbf{N} : \gcd(2^m - 1, 2^n - 1) = 1\} = \mathscr{O}_n. \end{aligned}$$

Thus, f is a homeomorphism between $\mathbf{X}$ and $\mathbf{Y}$. Therefore, $\mathbf{X}$ is homeomorphic to $\mathbf{Y}$.

Corollary 1. *The space $\mathbf{Y}$ does not satisfy the* T_0 *separation axiom. Furthermore, it is hyperconnected, ultraconnected, and therefore connected, locally connected, path-connected, normal, limit point compact, and pseudocompact.*

Since **Y** is hyperconnected, the closure of any of its open sets is **M**. On the other hand, it is well known that in any topological space, the closure of its closed sets is the same set. Therefore, so far we know what the closure of open and closed sets in **Y** is, however, we can say more. The following corollary gives us information about the closure of singleton sets in **Y**.

Corollary 2. *Let* $2^n - 1 \in \mathbf{M}$. *Then,* $\mathbf{Cl_Y}(\{2^n - 1\}) = \bigcap_{\substack{p\in\mathbf{P}\\ p|n}} \{2^{kp} - 1 : k \in \mathbf{N}\}$.

Proof. From Theorem 1 part 2 and the fact that f is a homeomorphism, we have:

$$\begin{aligned}\mathbf{Cl_Y}(\{2^n - 1\}) = \mathbf{Cl_Y}(\{f(n)\}) = f(\mathbf{Cl_X}(\{n\})) &= f\left(\bigcap_{\substack{p\in\mathbf{P}\\ p|n}} p\mathbf{N}\right)\\ &= \bigcap_{\substack{p\in\mathbf{P}\\ p|n}} f(p\mathbf{N})\\ &= \bigcap_{\substack{p\in\mathbf{P}\\ p|n}} \{2^{kp} - 1 : k \in \mathbf{N}\}.\end{aligned}$$

3 Topology $\tau_{\mathbf{M}}$: Characterization of the Conjecture of the Infinitude of Mersenne Primes

Theorem 2 can be used to characterizes the conjecture of the infinitude of the Mersenne primes with respect to the space **X**. We can do the same in the space **Y**, which is more convenient than **X**, given that unlike **X**, **Y** retains only the characteristics and properties of Mersenne numbers, as it precisely deals with numbers of the form $2^n - 1$. Furthermore, we can refine this characterization. With that said, the following theorem characterizes the conjecture of the infinitude of Mersenne primes in **Y**.

Theorem 5 (Characterization of the Conjecture of the Infinitude of Mersenne Primes). *Let* $\mathbf{M}_{(\mathbf{P})} := \{2^p - 1 : p \in \mathbf{P}\}$. *Then,* **MP** *is infinite if and only if it is dense in* $\mathbf{M}_{(\mathbf{P})}$ *with the subtopology* $\tau_{\mathbf{M}_{(\mathbf{P})}} := \{\mathscr{O} \cap \mathbf{M}_{(\mathbf{P})} : \mathscr{O} \in \tau_{\mathbf{M}}\}$.

Proof. Given that $f(\mathbf{P}) = \mathbf{M}_{(\mathbf{P})}$ and $f(\mathbf{N}) = \mathbf{M}$, we conclude that $\mathbf{M}_{(\mathbf{P})}$ is dense in **Y** since **P** is dense in **N**. Moreover, by Theorem 2, through f, we can deduce that for any subset A of $\mathbf{M}_{(\mathbf{P})}$ is infinite if and only if A is dense in **Y**. Take $A = \mathbf{MP}$. Then, by the transitivity of density, if **MP** is dense in $(\mathbf{M}_{(\mathbf{P})}, \tau_{\mathbf{M}_{(\mathbf{P})}})$, it is dense in **Y** and therefore infinite. On the other hand, if **MP** is infinite, it is clear that it is dense in $(\mathbf{M}_{(\mathbf{P})}, \tau_{\mathbf{M}_{(\mathbf{P})}})$.

The following proposition is generally satisfied.

Theorem 6. *Let $A \subset \mathbf{M}_{(\mathbf{P})}$. Then, A is infinite if and only if it is dense in* **Y**, *equivalently, dense in* **MP**.

4 Concluding Remarks

Finally, we present some remarks that pave the way for future work involving the space **Y**.

1. The space **M** is a topological copy of **X**. In this sense, for future work, one can explore topological properties in **X** (a more well-known and comfortable space to work with) and then transfer them to the space **Y** in order to obtain new properties about Mersenne numbers or provide topological proofs of already known properties of Mersenne numbers.
2. With the space **Y**, a topological proof of the infinitude of prime numbers can be obtained, similar to the one presented in [5]. In fact, there are infinitely many prime numbers if and only if **MP** is dense in **Y**.

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