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A generalization of the Zariouh's property (gaz) through local spectral theory

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Abstract In this work, we present for the first time an in-depth study of the relationship between the upper semi-B-Fredholm spectrum and the left Drazin spectrum. This connection leads to the definition of a new spectral property, denoted as $(ggaz)$, which generalizes the previously studied property (gaz). Through the framework of local spectral theory, we derive several characterizations of operators that satisfy the $(ggaz)$ property. Moreover, we demonstrate that the set of operators fulfilling this property constitutes a Banach space, highlighting the structural significance of $(ggaz)$ in operator theory.

Key words: Property (gaz), Property $(ggaz)$, semi-B-Fredholm operator, Left-Drazin Operator, $SVEP$.

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1 Introduction

The theory of Fredholm-type operators began around 1903 in a paper published by Erik Ivar Fredholm [2], in order to solve systems of differential or integro-differential equations. Spectral Theory arises when the system and its solution are defined in an infinite-dimensional space, where traditional methods are no longer applicable. Thus, the concept of an operator's spectrum T is introduced, and it is defined as $\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}$. Here, $\lambda I - T$ represents the operator modeling the system. Among the classical foundations of Spectral Theory are theorems of Browder-type and Weyl's theorems [1, Chapters 5, 6].

In operator theory, the classification of operators based on their spectral properties plays a central role in understanding the structure of the spectrum. Operators such as normal, Weyl-type, and Browder-type are classified to describe distinct regions of the spectrum, with various spectral properties defined to capture the behavior of these operators within their respective spectral subsets [1], [21]. The systematic development of these spectral properties has not only advanced the theoretical landscape of operator theory but has also fostered applications in areas such as data science [7] and artificial intelligence [6].

However, the Fredholm-type spectrum has received comparatively little attention. The majority of spectral properties introduced in [1] and [21] address spectra that define specific operator classes but do not fully engage with Fredholm-type operators. Only recently, in [15], it has any substantial analysis been conducted on Fredholm-type spectra, indicating a gap in the literature that warrants further exploration.

The study of Fredholm-type spectra is still in its early stages and remains innovative, particularly within the context of infinite-dimensional Banach spaces. The complexities inherent in such spaces necessitate new theoretical frameworks and methods, which presents an exciting frontier for research.

In this paper, we explore the relationship between the upper Berkani-Fredholm spectrum and the left Drazin spectrum, introducing a new spectral property denoted as $(ggaz)$, which extends the previously studied (gaz) property [5]. The $(ggaz)$ property offers a deeper insight into the spectral behavior of operators, and we provide several characterizations of operators that satisfy this property, as detailed in Section 3.

The results presented here have broader implications for future research. In particular, the $(ggaz)$ property can be extended to study the tensor product of operators that satisfy this property, potentially linking to the results found in [8, 9, 16, 20, 19]. Moreover, the analysis of $(ggaz)$ remains an open problem in areas such as perturbations and conjugate operators, as discussed in [4], paving the way for further advancements in local spectral theory.

2 Definitions and preliminary results

Let $L(X)$ be an algebra of bounded linear operators defined on an infinite-dimensional complex Banach space X. If $T \in L(X)$, then we denote by $\alpha(T)$ the dimension of the kernel $\ker T$, and by $\beta(T)$ the dimension of the range $R(T) := T(X)$, respectively.

Using $\alpha(T)$ and $\beta(T)$, we define Fredholm-type operators. The class of *upper semi-Fredholm* operators is defined as

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty, T(X) \text{ is closed}\}.$$

Similarly, the class of *lower semi-Fredholm* operators is defined by

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}.$$

Lower semi-Fredholm operators always have closed range because $\beta(T) < \infty$ implies that the range of T is closed.

The class of *Fredholm* operators is defined as $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$, while the class of semi-Fredholm operators is defined as $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$.

If $T \in \Phi_\pm(X)$, then the difference between $\alpha(T)$ and $\beta(T)$ is defined as a number that helps to express other classes of operators. This number is the index of T, and it is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$.

Using $\operatorname{ind}(T)$, the class of *Weyl operators* is defined as

$$W(X) := \{T \in \Phi(X) : \operatorname{ind}(T) = 0\},$$

the class of *upper semi-Weyl operators* is defined as

$$W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind}(T) \leq 0\},$$

and the class of *lower semi-Weyl operators* is defined as

$$W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind}(T) \geq 0\}.$$

On the other hand, we consider two additional numbers. Let $p := p(T)$ be the *ascent* of an operator $T \in L(X)$, defined as the smallest non-negative integer p such that $\ker T^p = \ker T^{p+1}$. If no such integer exists, we set $p(T) = \infty$. Similarly, let $q := q(T)$ be the *descent* of T, defined as the smallest non-negative integer q such that $T^q(X) = T^{q+1}(X)$. If no such integer exists, we set $q(T) = \infty$.

It is well known that if both $p(T)$ and $q(T)$ are finite, then $p(T) = q(T)$; see [1, Chapter 1]. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent; see [18, Proposition 50.2].

Using $p(T)$ and $q(T)$, another class of operators is defined: the class of all *Browder operators*, defined as the set

$$B(X) := \{T \in \Phi(X) : p(T) = q(T) < \infty\},$$

the class of all *upper semi-Browder operators* is defined as

$$B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

and the class of all *lower semi-Browder operators* is defined as

$$B_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

Note that $B(X) \subseteq W(X)$, $B_+(X) \subseteq W_+(X)$, and $B_-(X) \subseteq W_-(X)$.

In the following, we denote by $\sigma(T)$ the *spectrum* of T, defined as

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\},$$

and by $\sigma_{\mathrm{a}}(T)$ the *approximate point spectrum*, defined as

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

An operator is said to be *bounded below* if it is injective and has closed range. The surjective spectrum of T is defined as

$$\sigma_s(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is onto}\}.$$

In similar manner, other operator spectra are defined, namely:

- Upper semi-Fredholm spectrum: $\sigma_{uf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+(X)\}$.
- Upper semi B-Fredholm spectrum: $\sigma_{ubf}(T)$.
- Lower semi B-Fredholm spectrum: $\sigma_{lbf}(T)$.
- Approximate point spectrum: $\sigma_a(T)$.
- Weyl spectrum: $\sigma_w(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}$.
- Upper semi-Weyl spectrum: $\sigma_{uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\}$.
- Upper semi B-Weyl spectrum: $\sigma_{ubw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl}\}$.
- Upper semi-Browder spectrum: $\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\}$.

Semi-Fredholm operators have been generalized by Berkani ([12], [14], and [13]) in the following way: for every $T \in L(X)$ and a nonnegative integer n, let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$, viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). An operator $T \in L(X)$ is said to be *semi-B-Fredholm* (resp. *B-Fredholm*, *upper semi-B-Fredholm*, *lower semi-B-Fredholm*) if for some integer $n \geq 0$, the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case, $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ ([14]) with the same index as $T_{[n]}$. This allows us to define the index of a semi-B-Fredholm operator as $\operatorname{ind} T = \operatorname{ind} T_{[n]}$.

A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi-B-Fredholm*, *upper semi-B-Weyl*, *lower semi-B-Weyl*) if for some integer $n \geq 0$, the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Fredholm, upper semi-Weyl, lower semi-Weyl).

The *B-Weyl spectrum* is defined by

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\},$$

and the *upper semi-B-Weyl spectrum* of T is defined by

$$\sigma_{\mathrm{ubw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-B-Weyl}\}.$$

Analogously, the *upper semi-B-Fredholm spectrum* of T is defined by

$$\sigma_{\mathrm{ubf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-B-Fredholm}\}.$$

The numbers $p(T)$ and $q(T)$ allow the definition of a class of operators of Drazin type, which are considered in several studies within the theory of Fredholm operators. Thus, $T \in L(X)$ is said to be *Drazin invertible* (with a finite index) if $p(T) = q(T) < \infty$. It is said to be left Drazin invertible if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. Furthermore, $T \in L(X)$ is said to be right Drazin invertible if $q := q(T) < \infty$ and $T^q(X)$ is closed.

Note that $T \in L(X)$ is Drazin invertible if and only if T is both left Drazin invertible and right Drazin invertible.

If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{a}}(T)$, then λ is said to be a *left pole*. A left pole λ is said to have finite rank if $\alpha(\lambda I - T) < \infty$. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$, then λ is said to be a *right pole*. A right pole λ is said to have finite rank if $\beta(\lambda I - T) < \infty$.

The *Drazin spectrum* is defined as

$$\sigma_{\mathrm{d}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\},$$

the *left Drazin spectrum* is defined as

$$\sigma_{\mathrm{ld}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\},$$

while the *right Drazin spectrum* is defined as

$$\sigma_{\mathrm{rd}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible}\}.$$

Evidently, $\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{ld}}(T) \cup \sigma_{\mathrm{rd}}(T)$, $\sigma_{\mathrm{ubw}}(T) \subseteq \sigma_{\mathrm{ld}}(T)$, and $\sigma_{\mathrm{bw}}(T) \subseteq \sigma_{\mathrm{d}}(T)$.

Denote by $\Pi(T)$ and $\Pi_a(T)$ the set of all poles and the set of left poles of T, respectively. Clearly, $\Pi(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T)$ and $\Pi_a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$. Note that $\Pi_a(T) \subseteq \operatorname{iso}\sigma_a(T)$, where $\operatorname{iso}\sigma_a(T)$ denotes the set of isolated points of $\sigma_a(T)$.

In fact, if $\lambda_0 \in \Pi_a(T)$, then $\lambda I - T$ is left Drazin invertible and hence $p(\lambda_0 I - T) < \infty$. Since $\lambda I - T$ has a uniform topological descent (see [17] for definition and details), it follows from [17, Corollary 4.8] that $\lambda I - T$ is bounded below in a punctured disc centered at λ_0. Define $p_{00}^a(T) := \sigma_a(T) \setminus \sigma_{ub}(T)$ and $p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$; obviously, $p_{00}^a(T) \subseteq \Pi_a(T)$ and $p_{00}(T) \subseteq \Pi(T)$ for every $T \in L(X)$.

The theory of Fredholm-type operators is highly correlated with a property called SVEP. An operator $T \in L(X)$ is said to have *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to X$ that verifies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. An operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$, and both T and T^* have SVEP at the isolated points of the spectrum, where T^* denotes the dual operator, i.e., $T^*(\varphi) = \varphi \circ T$ for every φ in the dual space $X^* = L(X, \mathbb{C})$.

Remark 2.1 Note that

$$p(\lambda I - T) < \infty \iff T \textit{ has SVEP at } \lambda.$$

Moreover, from the definition of localized SVEP, we easily obtain that if

$$\sigma_{\mathrm{a}}(T) \textit{ does not cluster at } \lambda,$$

then T has SVEP at λ. These implications are equivalent when $\lambda I - T$ is an operator of Fredholm type. See [1, Chapter 2].

Note that T has SVEP at every isolated point of the spectrum, also in $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

Last but not least, we consider for $T \in L(X)$ the set:

$$\Xi(T) = \{\lambda \in \mathbb{C} : T \text{ does not possess the SVEP at } \lambda\}.$$

Note that $\Xi(T)$ is contained in the interior of the spectrum; according to the classical identity theorem for analytic functions, it follows that $\Xi(T)$ is open.

3 The generalized property $(ggaz)$

This section introduces a property that extends the scope of the existing property (gaz). This new property puts forward the idea that $\sigma_{ubf}(T) = \sigma_{ld}(T)$, which has not been considered so far. This idea is discussed in detail throughout the article, generating an analysis and relevance of the property.

For $T \in L(X)$, we define:

$$\Delta_a^g(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubf}}(T), \quad \Delta_1^g(T) := \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T), \text{ and } \quad \Delta^g(T) := \sigma(T) \setminus \sigma_{\mathrm{ubf}}(T).$$

Since $\sigma_{\mathrm{ubf}}(T) \subseteq \sigma_{\mathrm{ubw}}(T) \subseteq \sigma_{\mathrm{ld}}(T)$, we have:

$$\Pi_a(T) \subseteq \Delta_a^g(T) \subseteq \Delta_1^g(T) \subseteq \Delta^g(T).$$

Recall that $T \in L(X)$ is said to verify property (gaz) if $\Delta_1^g(T) = \Pi_a(T)$. Now, we define a generalization of the property (gaz).

Definition 3.1 *Let $T \in L(X)$. T is said to verify* generalized property $(ggaz)$ *if $\Delta^g(T) = \Pi_a(T)$.*

For $T \in L(X)$, the property $(ggaz)$ implies the property (gaz), but not vice versa. These properties are equivalent if $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ubf}}(T)$, or equivalently if $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uf}}(T)$. We establish this as a theorem.

Theorem 3.2 *$T \in L(X)$ verifies property $(ggaz)$ if and only if T verifies property (gaz) and $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uf}}(T)$.*

The above theorem and the following example show that property $(ggaz)$ generalizes property (gaz).

Example 3.3 *Let $X = \ell^2(\mathbb{N})$ and let T be the unilateral left shift defined as:*

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots), \text{ for all } (x_n) \in \ell^2(\mathbb{N}).$$

It is known that the upper semi-Weyl spectrum is the unit disc $\mathbf{D}(0,1)$. Also, by [15, Example 2.3], its upper semi-Fredholm spectrum is the unit circle $\Gamma(0,1)$. By Theorem 3.2, it follows that T does not verify property $(ggaz)$. However, by [5, Example 3.8], it turns out that T verifies property (gaz).

The following is an example of the application of Theorem 3.8.

Example 3.4 *An algebraic operator has a finite spectrum, so its dual has SVEP. Hence, an algebraic operator verifies property (gaz). Additionally, the accumulation points of the approximate point spectrum are empty, which is obvious, implying that $\sigma_{uw}(T) = \sigma_{uf}(T)$.*

In general, let $T \in L(X)$ and $\lambda \notin \sigma_{uf}(T) \cup Acc(\sigma_a(T))$. Then $\lambda \in \sigma_{uf}(T)^c \cap iso(\sigma_a(T))$; hence, $\lambda I - T$ is an upper semi-Fredholm operator and has SVEP at λ, giving that $p(\lambda I - T) < +\infty$, and thus $\lambda \notin \sigma_{uw}(T)$. We deduce that

$$\sigma_{uw}(T) \subseteq \sigma_{uf}(T) \cup Acc(\sigma_a(T)).$$

Therefore, if $Acc(\sigma_a(T)) = \emptyset$, then $\sigma_{uw}(T) = \sigma_{uf}(T)$.

By Theorem 3.2, it turns out that each algebraic operator verifies property $(ggaz)$.

Corollary 3.5 *Every operator $T \in L(X)$ with a finite spectrum verifies property $(ggaz)$.*

The SVEP is insufficient for operators to verify the property $(ggaz)$, as illustrated in the example below.

Example 3.6 *Let R denote the classical right shift in the Hilbert space $\ell_2(\mathbb{N})$, defined as*

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \quad \text{for all } x = (x_k)_{k \in \mathbb{N}} \in \ell_2(\mathbb{N}).$$

It is known that R has SVEP. However, the property $(ggaz)$ fails for R, since $\sigma(R) = \mathbf{D}(0,1)$, where $\mathbf{D}(0,1)$ is the closed disc in $\mathbb{C}$, and $\sigma_{\mathrm{a}}(R) = \partial\mathbf{D}(0,1)$.

Consider the sets

$$\Delta_+(T) := \sigma(T) \setminus \sigma_{uw}(T), \quad \delta_{uf}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{uf}(T).$$

Recently, studies have been done linking these sets with $p_{00}^a(T)$ in [11] and [10], throughout the properties (az) and (bz), respectively.

Definition 3.7 *Let $T \in L(X)$:*
1) If $\Delta_+(T) = p_{00}^a(T)$, we say that T verifies (az).
2) If $\delta_{uf}(T) = p_{00}^a(T)$, we say that T verifies (bz).

The properties (az) and (bz) together become the property $(ggaz)$. This is demonstrated by the following result.

Theorem 3.8 *Let $T \in L(X)$. Then T verifies $(ggaz)$ if and only if T verifies (az) and (bz).*

Proof. $\Rightarrow$) Let T verify the property $(ggaz)$. Then, by Theorem 3.2, T verifies the property (gaz) and $\sigma_{uf}(T) = \sigma_{uw}(T)$. Since T verifies (az), we have $\Delta_+(T) = p_{00}^a(T)$ (see [5, Theorem 3.5]), which implies that $\sigma_{ub}(T) = \sigma_{uw}(T)$. Hence, $\sigma_{ub}(T) = \sigma_{uf}(T)$, giving the property (bz) for T. Moreover, since (gaz) is equivalent to (az) (see [5]), we obtain the result.

$\Leftarrow$) Conversely, suppose that T verifies properties (az) and (bz). Then T verifies (gaz) since (az) is equivalent to (gaz). It is known that $\sigma_{uf}(T) \subseteq \sigma_{uw}(T) \subseteq \sigma_{ub}(T)$. Since T verifies (bz), we have $\sigma_{uf}(T) = \sigma_{ub}(T)$ (see [**?**, Theorem 3.5]), implying that $\sigma_{uf}(T) = \sigma_{uw}(T)$. Thus, by Theorem 3.2, the result follows.

Since $L(X)$ is a Banach algebra, Theorem 3.8 allows us to establish that the set of operators that verify the property $(ggaz)$ is a Banach space, which is achieved by proving that it is closed in $L(X)$, as shown below.

Theorem 3.9 *Let $T \in L(X)$ and $T_n \in L(X)$ for each $n \geq 1$, such that $\lim \|T - T_n\| = 0$ when $n \to \infty$. If for each $n \geq 1$, T_n verifies the property $(ggaz)$, then T verifies the property $(ggaz)$.*

Proof. By Theorem 3.8, we have that each T_n verifies properties (az) and (bz). Thus, by [11, Theorem 9], T verifies the property (az), and by [10, Theorem 3.11], T verifies the property (bz). Therefore, by Theorem 3.8, we conclude that T verifies the property $(ggaz)$.

Corollary 3.10 *The set of operators satisfying the property $(ggaz)$ is a Banach space.*

Example 3.11 *Let $X = \ell^2(\mathbb{N})$, and let $(a_n) \in \mathbb{C}$ be a sequence of non-zero complex numbers converging to 0.*

Define the sequence of operators as $T_n(x) = (a_1x_1, \ldots, a_nx_n, 0, 0, \ldots)$, for each $x = (x_n) \in X$. Note that $\lim \|T - T_n\| = 0$ when $n \to \infty$, where

$$T(x) = (a_1x_1, a_2x_2, \ldots, a_nx_n, \ldots) \quad \forall x = (x_n) \in X.$$

On the other hand, for all $n \in \mathbb{N}$, it turns out that $\sigma(T_n) = \{a_1, \ldots, a_n\}$. Thus, by Corollary 3.5, we have that for all positive integers n, T_n verifies the property $(ggaz)$. Therefore, by Theorem 3.9, the limit T verifies the property $(ggaz)$.

In the following, we characterize the operators that verify property $(ggaz)$. In particular, if the approximate point spectrum differs from the spectrum, the operator does not verify property $(ggaz)$.

Theorem 3.12 *$T \in L(X)$ verifies property $(ggaz)$ if and only if $\sigma_{\mathrm{a}}(T) = \sigma(T)$ and T verifies property (bz).*

Proof. $(\Rightarrow)$ Note that T verifies property (gaz), so $\sigma(T) = \sigma_a(T)$. Also, by Theorem 3.2, it follows that $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uf}}(T)$. Since the properties (gaz) and (az) are equivalent, we have $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$. Therefore, we obtain that $\sigma_{\mathrm{uf}}(T) = \sigma_{\mathrm{ub}}(T)$, i.e., T verifies property (bz).

$(\Leftarrow)$ Note that $\sigma_{\mathrm{uf}}(T) \subseteq \sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$. Thus, since T verifies property (bz), it follows that $\sigma_{\mathrm{uf}}(T) = \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$. Also, by hypothesis, $\sigma(T) = \sigma_{\mathrm{a}}(T)$, which implies that T verifies property (az), or equivalently, property (gaz). Therefore, by Theorem 3.2, we conclude that T verifies property $(ggaz)$. ∎

We denote by $\mathscr{H}(\sigma(T))$ the set of all analytic functions defined in an open neighborhood of the spectrum $\sigma(T)$. For $f \in \mathscr{H}(\sigma(T))$, we consider $f(T)$ as in the classical Riesz functional calculus. The spectral mapping theorem is valid for $\sigma(T)$ and $\sigma_a(T)$, i.e., $\sigma(f(T)) = f(\sigma(T))$ and $\sigma_{\mathrm{a}}(f(T)) = f(\sigma_{\mathrm{a}}(T))$, where $f \in \mathscr{H}(\sigma(T))$. Additionally, by [10, Theorem 3.7], it turns out that $f(T)$ verifies property (bz) if T verifies property (bz). Therefore, we obtain the following result from Theorem 3.12.

Corollary 3.13 *Let $T \in L(X)$ and $f \in \mathscr{H}(\sigma(T))$. If T verifies property $(ggaz)$, then $f(T)$ verifies property $(ggaz)$.*

The following characterization for the operators that verify property $(ggaz)$ shows the spectral structure they must have.

Theorem 3.14 *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *T has property $(ggaz)$;*

(ii) *$\Delta^g(T) \subseteq iso\,\sigma_{\mathrm{a}}(T)$;*

(iii) *$\Delta^g(T) \subseteq \partial\sigma_{\mathrm{a}}(T)$, where $\partial\sigma_{\mathrm{a}}(T)$ is the boundary of $\sigma_{\mathrm{a}}(T)$;*

(iv) *$int\,\Delta^g(T) = \emptyset$;*

(v) *$\sigma(T) = \sigma_{\mathrm{ubf}}(T) \cup \partial\sigma_{\mathrm{a}}(T)$;*

(vi) *$\sigma(T) = \sigma_{\mathrm{ubf}}(T) \cup iso\,\sigma_{\mathrm{a}}(T)$.*

(vii) *$\sigma_{\mathrm{ubf}}(T) = \sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$.*

Proof. (i) $\Rightarrow$ (ii) This is because $\Pi_a(T) \subseteq \operatorname{iso}\sigma_a(T)$.

(ii) $\Rightarrow$ (iii) Clear, since $\operatorname{iso}\sigma_{\mathrm{a}}(T) \subseteq \partial\sigma_{\mathrm{a}}(T)$.

(iii) $\Rightarrow$ (iv) Clear.

(iv) $\Rightarrow$ (i) Let $\lambda_0 \in \Delta^g(T)$. If T does not have SVEP for some $\lambda_0 \in \Delta^g(T)$, then T does not have SVEP for all $\lambda \in D(\lambda_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, because $\Xi(T)$ is an open set.

If $\lambda_0 I - T$ is upper semi-B-Fredholm, then by [1, Theorem 1.117] there exists an open disc $D(0,\varepsilon)$ centered at 0 such that $\lambda I - T$ is upper semi-Fredholm for all $\lambda \in D(0,\varepsilon) \setminus \{0\}$. Since each upper semi-Fredholm operator is an upper semi-B-Fredholm operator, we deduce that the set of upper semi-B-Fredholm operators is an open set. So, we consider an open disk $D(\lambda_0, \varepsilon_1)$ for $\lambda_0 I - T$.

Now, if $\varepsilon_2 := \min\{\varepsilon_0, \varepsilon_1\}$, then for every $\lambda \in D(\lambda_0, \varepsilon_2)$, $\lambda I - T$ does not have SVEP (thereby $\lambda \in \sigma(T)$) and is an upper semi-B-Fredholm operator. Consequently, $D(\lambda_0, \varepsilon_2) \subseteq \Delta^g(T)$, but this is impossible.

Therefore, T has SVEP at λ_0, and by Remark 2.1, it turns out that $p(\lambda_0 I - T) < \infty$. In view of [1, Corollary 1.83] and [1, Theorem 1.81], we deduce that $\lambda \in \Pi_a(T)$.

Therefore, $\Delta^g(T) \subseteq \Pi_a(T)$ and T has property $(ggaz)$.

(iv) $\Leftrightarrow$ (v) Note that (v) $\Rightarrow$ (iv), also (iv) $\Rightarrow$ (iii), and that (iii) $\Rightarrow$ (v).

(iv) $\Leftrightarrow$ (vi) Note that (vi) $\Rightarrow$ (iv), also (iv) $\Rightarrow$ (ii), and that (ii) $\Rightarrow$ (vi).

(i) $\Leftrightarrow$ (vii) Directly. Note that T verifies property (gaz), whereby $\sigma(T) = \sigma_a(T)$. Since T verifies property $(ggaz)$, we obtain that $\sigma_{\mathrm{ubf}}(T) = \sigma_{\mathrm{ld}}(T)$. The result is obtained by [5, Theorem 3.3].

Conversely, since $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$, it follows that $\sigma(T) = \sigma_a(T)$. Given the hypothesis, we have that T verifies property $(ggaz)$. ∎

Finally, in terms of connection, we establish a sufficient condition for an operator to verify the property $(ggaz)$.

Theorem 3.15 *Let $T \in L(X)$. If $\rho_{uf}(T)$ and $\rho_{uw}(T)$ are connected, then T verifies the property $(ggaz)$.*

Proof. T verifies the SVEP for all $\lambda \in \rho(T)$. Now, if Ω is the unique component of $\rho_{uf}(T)$, then by [3, Theorem 3.36], T has the SVEP for all $\lambda \in \Omega$. We consider $\lambda \notin \sigma_{uf}(T)$; in this way, $\lambda \in \rho_{uf}(T)$, hence, $\lambda \in \Omega$. Consequently, T has the SVEP at λ. Since $\lambda \notin \sigma_{uf}(T)$, it follows that $p(\lambda I - T) < +\infty$, implying that $\lambda \notin \sigma_{uw}(T)$. Thus, we deduce that $\sigma_{uf}(T) = \sigma_{uw}(T)$. On the other hand, by [5, Theorem 3.12], T verifies property (gaz). Therefore, T verifies property $(ggaz)$.

4 Conclusions

The property $(ggaz)$ collects operators that are upper semi-B-Fredholm, are not injective, and have finite ascent. If an operator $T \in L(X)$ does not verify the property $(ggaz)$, then:

1. $\sigma_{uf}(T) \neq \sigma_{uw}(T)$, or T does not verify the property (gaz), see Theorem 3.2.

2. T does not verify the property (az), or does not verify the property (bz), see Theorem 3.8.
3. $\operatorname{int}\Delta^g(T) \neq \emptyset$, see Theorem 3.14.
4. $\rho_{uf}(T)$, or $\rho_{uw}(T)$ are not connected.

On the other hand, the property $(ggaz)$ is satisfied under the Riesz calculus, see Corollary 3.13. Also, the set of operators verifying the property $(ggaz)$ forms a closed set in $L(X)$; therefore, it is a Banach space in $L(X)$, see Theorem 3.9.

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