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Geometrical interpretation of the circle equation in the complex plane using the equivalent real-valued function

Interpretación geométrica de la ecuación del círculo en el plano complejo usando la función de valor real equivalente

Fernando Gómez-Villarraga

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Abstract If tangent circles are drawn to a circle with center on the x-axis, the roots of the tangent circles that do not touch the x-axis and the root of the tangent circle that only touch the x-axis at one point are located on a circle in the complex plane. A function of a circle in the complex plane is obtained. The complex function represents the complex roots (discriminant less than zero) and the unique real solution (discriminant is equal to zero) of the first (or second) tangent circles to the real-valued function that represents the superior (or inferior) part of a circle with center on the x-axis.
Keywords complex function, real-valued function, tangent circle.

Resumen Si a un círculo con centro en el eje x se le trazan círculos tangentes, las raíces de los círculos tangentes que no tocan el eje x y la raíz del círculo tangente que solo toca el eje x en un punto se ubican sobre un círculo en el plano complejo. Una función de un círculo en el plano complejo es obtenida. La función compleja representa las raíces complejas (discriminante menor que cero) y la única solución real (discriminante igual a cero) de los primeros (o segundos) círculos tangentes a la función de valor real que representa la parte superior (o inferior) de un círculo con centro en el eje x.
Palabras Claves círculo tangente, función compleja, función de valor real.

1 Introduction

An application of the tangent conic sections to the graph of a function is presented (Gómez-Villarraga, 2021). The straight line is the simplest tangent curve to

Fernando Gómez-Villarraga, PhD.
Independent researcher, Heinigstraße 15, 67059 Ludwigshafen am Rhein, Germany, e-mail: ferchogomezv@gmail.com, https://orcid.org/0000-0001-5860-6951

the graph of a function at a point P. Tangent conic sections to the graph of a function can also be determined. This can be seen as curves tangent to other curve. The first (or second) tangent circles to the graph of a function are determined using the equations for the first (or second) tangent ellipses where the semi-major axis a and the semi-minor axis b are equal to the radius r (Gómez-Villarraga, 2021); (Larson & Edwards, 2008); (Leithold, 1998); (Stewart, 2015); (Strang, 2010); (Tan, 2010); (Thomas, Weir, & Hass, 2013).

The complex roots (discriminant less than zero) and the unique real solution (discriminant is equal to zero) of the first (or second) tangent circles to the real-valued function that represents the superior (or inferior) part of a circle with center on the x-axis are on a circle in the complex plane. In other words, if tangent circles are drawn to a circle with center on the x-axis, the roots of the tangent circles that do not touch the x-axis and the root of the tangent circle that only touch the x-axis at one point are located on a circle in the complex plane (Howie, 2003); (Newcomb, 1885); (Swokowski, 1979); (Swokowski & Cole, 2012); (Zill & Shanahan, 2009).

A geometrical connection between a circle equation in the complex plane and the equivalent real-valued function is found.

2 Geometrical interpretation of the circle equation in the complex plane using the equivalent real-valued function

The real-valued function of a circle with center at (h, k) and radius $r > 0$ is given by:

$$(x-h)^2 + (y-k)^2 = r^2 \tag{1}$$

(Leithold, 1998, p. 1173)

Solving for y in equation 1

$$y = \pm\sqrt{r^2 - (x-h)^2} + k \quad \text{where } h - r \leq x \leq h + r \tag{2}$$

Two functions are obtained from the equation 2:

$$y = f(x) = \sqrt{r^2 - (x-h)^2} + k \quad \text{where } h - r \leq x \leq h + r \tag{3}$$

And:

$$y = f(x) = -\sqrt{r^2 - (x-h)^2} + k \quad \text{where } h - r \leq x \leq h + r \tag{4}$$

The functions 3 and 4 are plotted in Figure 1:

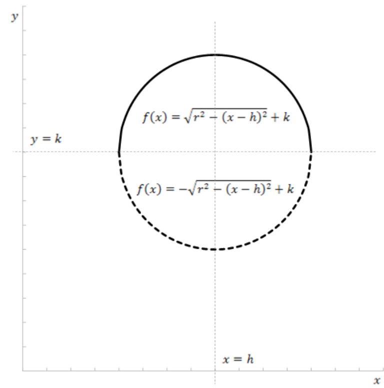
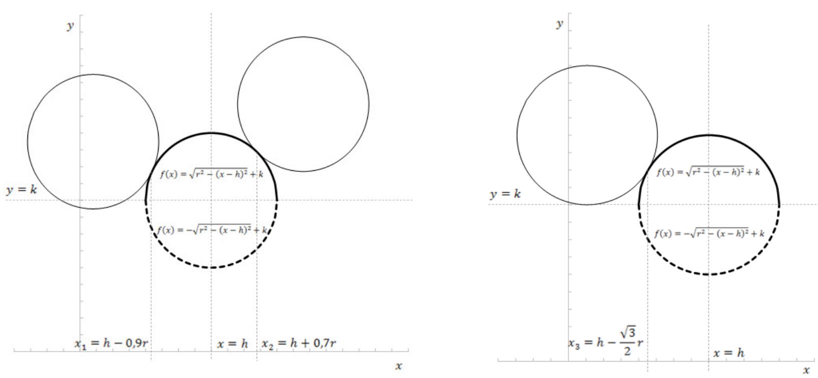


Figure 1: Graph of the function $f(x) = \sqrt{r^2 - (x-h)^2} + k$ and $f(x) = -\sqrt{r^2 - (x-h)^2} + k$ (Swokowski, 1979; Swokowski & Cole, 2012)
Source: Own creation

First tangent circles of radius r (the radius of the tangent circles is the same as the original circle) to the function $f(x) = \sqrt{r^2 - (x-h)^2} + k$ at $x_1 = h - 0.9r$, $x_2 = h + 0.7r$ and $x_3 = h - \frac{\sqrt{3}}{2}r$ are plotted in Figure 2. The first tangent circles are determined using the equations for the first tangent ellipses to the graph of a function where the semi-major axis a and the semi-minor axis b are equal to r. The graphs are determined using the parametric equations (Gómez-Villarraga, 2021).



(a) $x_1 = h - 0.9r$ and $x_2 = h + 0.7r$ **(b)** $x_3 = h - \frac{\sqrt{3}}{2}r$

Figure 2: First tangent circles of radius r to the function $f(x) = \sqrt{r^2 - (x-h)^2} + k$ at different points.
Source: Own creation

The first tangent circle to the function $f(x) = \sqrt{r^2 - (x-h)^2} + k$ at $x_1 = h - 0.9r$ crosses the line $y = k$ at two points. The first tangent circle at $x_2 = h + 0.7r$ does not cross the line $y = k$. The first tangent circle to the function $f(x) = \sqrt{r^2 - (x-h)^2} + k$ at $x_3 = h - \frac{\sqrt{3}}{2}r$ crosses the line $y = k$ at one point. Several first tangent circles to the functions $f(x) = \sqrt{r^2 - (x-h)^2} + k$ and $f(x) = -\sqrt{r^2 - (x-h)^2} + k$ are plotted in Figure 3. The first tangent circles to the function $f(x) = \sqrt{r^2 - (x-h)^2} + k$ generate a crown-like graph, there some circles cross the line $y = k$ and others do not. The first tangent circles to the function $f(x) = -\sqrt{r^2 - (x-h)^2} + k$ coincide with the original circle.

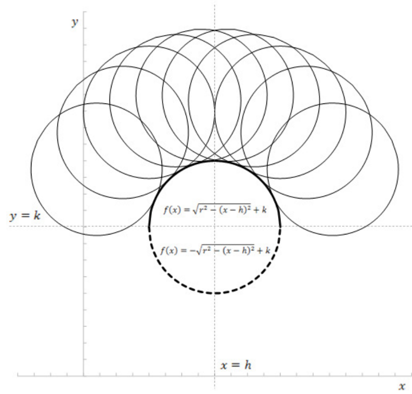


Figure 3: Several first tangent circles of radius r to the functions $f(x) = \sqrt{r^2 - (x-h)^2} + k$ and $f(x) = -\sqrt{r^2 - (x-h)^2} + k$.
Source: Own creation

The equation for the first tangent circle to the graph of a function $y = f(x)$ at a point $(x_i, f(x_i))$ can be written using the general second-degree equation $A_i x^2 + B_i xy + C_i y^2 + D_i x + E_i y + F_i = 0$ (Gómez-Villarraga, 2021). The roots are calculated considering $y = 0$ in the equation. Thus:

$$A_i x^2 + D_i x + F_i = 0 \tag{5}$$

Where:

$$A_i = \frac{1}{r^2} \tag{6}$$

(Gómez-Villarraga, 2021, p. 36)

$$D_i = \frac{2f'(x_i)}{r\sqrt{1 + [f'(x_i)]^2}} - \frac{2x_i}{r^2} \tag{7}$$

(Gómez-Villarraga, 2021, p. 36)

$$F_i = \frac{x_i^2 + [f(x_i)]^2}{r^2} - \frac{2[f'(x_i)x_i - f(x_i)]}{r\sqrt{1 + [f'(x_i)]^2}} \tag{8}$$

(Gómez-Villarraga, 2021, p. 36)

The expressions for the coefficients A_i, D_i and F_i can be found in Gómez-Villarraga (2021). The quadratic equation has two solutions given by:

$$x_{i+} = \frac{-D_i + \sqrt{D_i^2 - 4A_iF_i}}{2A_i} \tag{9}$$

$$x_{i-} = \frac{-D_i - \sqrt{D_i^2 - 4A_iF_i}}{2A_i} \tag{10}$$

The discriminant $D_i^2 - 4A_iF_i$ of the of the quadratic equation can be calculated using the equations 6 and 8:

$$\begin{aligned} D_i^2 - 4A_iF_i = {} & \frac{4[f'(x_i)]^2}{r^2\{1 + [f'(x_i)^2]\}} - \frac{8x_if'(x_i)}{r^3\sqrt{1 + [f'(x_i)]^2}} + \frac{4x_i^2}{r^4} \\ & - \frac{4x_i^2}{r^4} - \frac{4[f(x_i)]^2}{r^4} + \frac{8[f'(x_i)x_i - f(x_i)]}{r^3\sqrt{1 + [f'(x_i)]^2}} \end{aligned} \tag{11}$$

Simplifying the equation 11:

$$D_i^2 - 4A_iF_i = \frac{4[f'(x_i)]^2}{r^2\{1 + [f'(x_i)^2]\}} - \frac{8f(x_i)}{r^3\sqrt{1 + [f'(x_i)]^2}} - \frac{4[f(x_i)]^2}{r^4} \tag{12}$$

The first tangent circles to the function $f(x) = \sqrt{r^2 - (x-h)^2} + k$ are considered. $x_i, f(x_i), f'(x_i)$ can be replaced by $x, f(x), f'(x)$ respectively in equation 12. $f(x)$ is $\sqrt{r^2 - (x-h)^2} + k$ and $f'(x)$ is given by $-\frac{x-h}{\sqrt{r^2-(x-h)^2}}$. Thus:

$$D_i^2 - 4A_iF_i = \frac{4\frac{(x-h)^2}{r^2-(x-h)^2}}{r^2\left\{1 + \frac{(x-h)^2}{r^2-(x-h)^2}\right\}} - \frac{8\sqrt{r^2-(x-h)^2} + 8k}{r^3\sqrt{1 + \frac{(x-h)^2}{r^2-(x-h)^2}}} - \frac{4[r^2 - (x-h)^2 + 2k\sqrt{r^2-(x-h)^2} + k^2]}{r^4} \tag{13}$$

Simplifying the equation 13:

$$D_i^2 - 4A_iF_i = \frac{4\frac{(x-h)^2}{r^2-(x-h)^2}}{r^2\left\{\frac{r^2-(x-h)^2+(x-h)^2}{r^2-(x-h)^2}\right\}} - \frac{8\sqrt{r^2-(x-h)^2} + 8k}{r^3\sqrt{\left\{\frac{r^2-(x-h)^2+(x-h)^2}{r^2-(x-h)^2}\right\}}} - \frac{4[r^2 - (x-h)^2 + 2k\sqrt{r^2-(x-h)^2} + k^2]}{r^4} \tag{14}$$

$$D_i^2 - 4A_iF_i = \frac{4\frac{(x-h)^2}{r^2-(x-h)^2}}{r^2\left\{\frac{r^2}{r^2-(x-h)^2}\right\}} - \frac{8\sqrt{r^2-(x-h)^2}+8k}{r^3\sqrt{\frac{r^2}{r^2-(x-h)^2}}} - \frac{4[r^2-(x-h)^2+2k\sqrt{r^2-(x-h)^2}+k^2]}{r^4} \tag{15}$$

$$\begin{aligned} D_i^2 - 4A_iF_i = {} & \frac{4(x-h)^2}{r^4} - \frac{8[r^2-(x-h)^2]}{r^4} - \frac{8k\sqrt{r^2-(x-h)^2}}{r^4} \\ & - \frac{4}{r^2} + \frac{4(x-h)^2}{r^4} - \frac{8k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{4k^2}{r^4} \end{aligned} \tag{16}$$

$$D_i^2 - 4A_iF_i = \frac{4(x-h)^2}{r^4} - \frac{8}{r^2} + \frac{8(x-h)^2}{r^4} - \frac{8k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{4}{r^2} + \frac{4(x-h)^2}{r^4} - \frac{8k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{4k^2}{r^4} \tag{17}$$

$$D_i^2 - 4A_iF_i = \frac{16(x-h)^2}{r^4} - \frac{16k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{12}{r^2} - \frac{4k^2}{r^4} \tag{18}$$

If the discriminant $D_i^2 - 4A_iF_i$ is less than 0 the quadratic equation has two imaginary solutions. If it's equal to 0, there is one real solution of multiplicity 2 Swokowski (1979); Swokowski & Cole (2012). Using the equation 18:

$$\frac{16(x-h)^2}{r^4} - \frac{16k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{12}{r^2} - \frac{4k^2}{r^4} \leq 0 \tag{19}$$

Ordering the equation 19

$$\frac{16(x-h)^2}{r^4} - \frac{16k\sqrt{r^2-(x-h)^2}}{r^4} \leq \frac{12}{r^2} + \frac{4k^2}{r^4} \tag{20}$$

If the center at (h, k) is on the x-axis ($k = 0$):

$$\frac{16(x-h)^2}{r^4} \leq \frac{12}{r^2} \tag{21}$$

$$(x-h)^2 \leq \frac{3r^2}{4} \tag{22}$$

$$|x-h| \leq \frac{\sqrt{3}}{2}r \tag{23}$$

$$-\frac{\sqrt{3}}{2}r \leq x-h \leq \frac{\sqrt{3}}{2}r \tag{24}$$

$$h - \frac{\sqrt{3}}{2}r \leq x \leq h + \frac{\sqrt{3}}{2}r \tag{25}$$

$x_i, f(x_i), f'(x_i)$ can also be replaced by $x, f(x), f'(x)$ in the expressions for $2A_i$ and $-D_i$. $f(x)$ is $\sqrt{r^2-(x-h)^2}+k$ and $f'(x)$ is given by $-\frac{x-h}{\sqrt{r^2-(x-h)^2}}$. Thus:

$$2A_i = \frac{2}{r^2} \tag{26}$$

$$-D_i = \frac{2x_i}{r^2} - \frac{2f'(x_i)}{r\sqrt{1+[f'(x_i)]^2}} \tag{27}$$

$$-D_i = \frac{2x}{r^2} + \frac{\frac{2(x-h)}{\sqrt{r^2-(x-h)^2}}}{r\sqrt{1+\frac{(x-h)^2}{r^2-(x-h)^2}}} \tag{28}$$

Simplifying the equation 28

$$-D_i = \frac{2x}{r^2} + \frac{\frac{2(x-h)}{\sqrt{r^2-(x-h)^2}}}{r\sqrt{\frac{r^2-(x-h)^2+(x-h)^2}{r^2-(x-h)^2}}} \tag{29}$$

$$-D_i = \frac{2x}{r^2} + \frac{\frac{2(x-h)}{\sqrt{r^2-(x-h)^2}}}{\frac{r^2}{\sqrt{r^2-(x-h)^2}}} \tag{30}$$

$$-D_i = \frac{2x}{r^2} + \frac{2(x-h)}{r^2} \tag{31}$$

The expressions for $-D_i$ (equation 31) $D_i^2 - 4A_iF_i$ (equation 18) and $2A_i$ (equation 26) are replaced in the equations 9 and 10:

$$x_{i+} = \frac{\frac{2x}{r^2} + \frac{2(x-h)}{r^2} + \sqrt{\frac{16(x-h)^2}{r^4} - \frac{16k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{12}{r^2} - \frac{4k^2}{r^4}}}{\frac{2}{r^2}} \tag{32}$$

$$x_{i+} = 2x - h + \sqrt{4(x-h)^2 - 4k\sqrt{r^2-(x-h)^2} - 3r^2 - k^2} \tag{33}$$

$$x_{i-} = \frac{\frac{2x}{r^2} + \frac{2(x-h)}{r^2} - \sqrt{\frac{16(x-h)^2}{r^4} - \frac{16k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{12}{r^2} - \frac{4k^2}{r^4}}}{\frac{2}{r^2}} \tag{34}$$

$$x_{i-} = 2x - h - \sqrt{4(x-h)^2 - 4k\sqrt{r^2-(x-h)^2} - 3r^2 - k^2} \tag{35}$$

If the center at (h, k) is on the x-axis ($k = 0$)

$$x_{i+} = 2x - h + \sqrt{4(x-h)^2 - 3r^2} \tag{36}$$

$$x_{i-} = 2x - h - \sqrt{4(x-h)^2 - 3r^2} \tag{37}$$

The imaginary unit i can be expressed explicitly in the equations 33 and 35, taking into account the complex number solutions of the quadratic equation (when the discriminant $D_i^2 - 4A_iF_i$ is less than 0):

$$x_{i+} = 2x - h + \sqrt{(-1)[4(x-h)^2 - 4k\sqrt{r^2-(x-h)^2} - 3r^2 - k^2]}\,i \qquad (38)$$

$$x_{i-} = 2x - h - \sqrt{(-1)[4(x-h)^2 - 4k\sqrt{r^2-(x-h)^2} - 3r^2 - k^2]}\,i \qquad (39)$$

Simplifying the equations 38 and 39:

$$x_{i+} = 2x - h + \sqrt{4k\sqrt{r^2-(x-h)^2} + 3r^2 + k^2 - 4(x-h)^2}\,i \qquad (40)$$

$$x_{i-} = 2x - h - \sqrt{4k\sqrt{r^2-(x-h)^2} + 3r^2 + k^2 - 4(x-h)^2}\,i \qquad (41)$$

Calculating the distance between the point given by the equation 40 and the point $h + ki$:

$$d=\sqrt{(2x-2h)^2+\left[\sqrt{4k\sqrt{r^2-(x-h)^2}+3r^2+k^2-4(x-h)^2}-k\right]^2} \qquad (42)$$

$$d=\sqrt{4x^2-8hx+4h^2+4k\sqrt{r^2-(x-h)^2}+3r^2+k^2-4x^2+8hx-4h^2-2k\sqrt{4k\sqrt{r^2-(x-h)^2}+3r^2+k^2-4(x-h)^2}+k^2} \qquad (43)$$

$$d = \sqrt{4k\sqrt{r^2-(x-h)^2} - 2k\sqrt{4k\sqrt{r^2-(x-h)^2} + 3r^2 + k^2 - 4(x-h)^2} + 2k^2 + 3r^2} \qquad (44)$$

Similarly, the distance between the point given by equation 41 and the point $h+ki$ can be calculated:

$$d = \sqrt{(2x-2h)^2 + \left[-\sqrt{4k\sqrt{r^2-(x-h)^2} + 3r^2 + k^2 - 4(x-h)^2} - k\right]^2} \qquad (45)$$

$$d = \sqrt{(2x-2h)^2 + \left[\sqrt{4k\sqrt{r^2-(x-h)^2} + 3r^2 + k^2 - 4(x-h)^2} + k\right]^2} \qquad (46)$$

$$d=\sqrt{4x^2-8hx+4h^2+4k\sqrt{r^2-(x-h)^2}+3r^2+k^2-4x^2+8hx-4h^2+2k\sqrt{4k\sqrt{r^2-(x-h)^2}+3r^2+k^2-4(x-h)^2}+k^2} \qquad (47)$$

$$d = \sqrt{4k\sqrt{r^2-(x-h)^2} + 2k\sqrt{4k\sqrt{r^2-(x-h)^2} + 3r^2 + k^2 - 4(x-h)^2} + 2k^2 + 3r^2} \qquad (48)$$

If the center of the circle at (h, k) is any point with $h \neq 0$ and $k \neq 0$, the distance d of the complex roots of the first tangent circles of radius r to the real-valued function $f(x) = \sqrt{r^2 - (x-h)^2} + k$ that represents the superior part of a circle with center at (h, k) and radius r is not constant (equations 44 and 48). So, the complex roots are not on a circle in the complex plane. But if the center of the circle at (h, k) is any point on the x-axis ($k = 0$), the distance d in equations 44 and 48 is:

$$d = 2\sqrt{3} \tag{49}$$

In this case the complex roots are on a circle in the complex plane. A complex function can be established using the complex solutions of the quadratic equation. Restating the equations 36 and 37 as a complex function:

$$f(z) = 2z - h \pm \sqrt{4(z-h)^2 - 3r^2} \tag{50}$$

Using the restriction in the domain given by equation 25:

$$h - \frac{\sqrt{3}}{2} r \leq Re(Z) \leq h + \frac{\sqrt{3}}{2} \quad and \quad Im(z) = 0 \tag{51}$$

The complex function given by the equation equation 50 with the domain given by the equation 51 is a circle in the complex plane with radius r $\sqrt{3}$ *(equation 49).The complex function represents the complex roots (discriminant less than zero) and the unique real solution (discriminant is equal to zero) of the first tangent circles of radius r to the real-valued function* $f(x) = \sqrt{r^2 - (x-h)^2}$ *that represents the superior part of a circle with center on the x-axis at* $(h, 0)$ *and radius r.*
As mentioned before, the first tangent circles to the function $f(x) = -\sqrt{r^2 - (x-h)^2} + k$ coincide with the original circle $x_i, f(x_i), f'(x_i)$ can be replaced by $x, f(x), f'(x)$ respectively in the equation 12. $f(x) = -\sqrt{r^2 - (x-h)^2} + k$ and $f'(x)$ is given by: $\frac{x-h}{\sqrt{r^2-(x-h)^2}}$. Thus:

$$D_i^2 - 4A_iF_i = \frac{4\frac{(x-h)^2}{r^2-(x-h)^2}}{r^2\left\{1 + \frac{(x-h)^2}{r^2-(x-h)^2}\right\}} + \frac{8\sqrt{r^2-(x-h)^2} - 8k}{r^3\sqrt{1 + \frac{(x-h)^2}{r^2-(x-h)^2}}} - \frac{4[r^2 - (x-h)^2 - 2k\sqrt{r^2-(x-h)^2} + k^2]}{r^4} \tag{52}$$

Simplifying the equation 52:

$$D_i^2 - 4A_iF_i = \frac{4\frac{(x-h)^2}{r^2-(x-h)^2}}{r^2\frac{r^2-(x-h)^2+(x-h)^2}{r^2-(x-h)^2}} + \frac{8\sqrt{r^2-(x-h)^2} - 8k}{r^3\sqrt{\frac{r^2-(x-h)^2+(x-h)^2}{r^2-(x-h)^2}}} - \frac{4[r^2 - (x-h)^2 - 2k\sqrt{r^2-(x-h)^2} + k^2]}{r^4} \tag{53}$$

$$D_i^2 - 4A_iF_i = \frac{4\frac{(x-h)^2}{r^2-(x-h)^2}}{r^2\frac{r^2}{r^2-(x-h)^2}} + \frac{8\sqrt{r^2-(x-h)^2} - 8k}{r^3\sqrt{\frac{r^2}{r^2-(x-h)^2}}} - \frac{4[r^2 - (x-h)^2 - 2k\sqrt{r^2-(x-h)^2} + k^2]}{r^4} \tag{54}$$

$$D_i^2 - 4A_iF_i = \frac{4(x-h)^2}{r^4} + \frac{8[r^2-(x-h)^2]}{r^4} - \frac{8k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{4}{r^2} + \frac{4(x-h)^2}{r^4} + \frac{8k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{4k^2}{r^4} \tag{55}$$

$$D_i^2 - 4A_iF_i = \frac{4(x-h)^2}{r^4} + \frac{8}{r^2} - \frac{8(x-h)^2}{r^4} - \frac{8k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{4}{r^2} + \frac{4(x-h)^2}{r^4} + \frac{8k\sqrt{r^2-(x-h)^2}}{r^4} - \frac{4k^2}{r^4} \tag{56}$$

$$D_i^2 - 4A_iF_i = \frac{4}{r^2} - \frac{4k^2}{r^4} \tag{57}$$

$x_i, f(x_i), f'(x_i)$ can also be replaced by $x, f(x), f'(x)$ in the expressions for $2A_i$ (equation 26) and $-D_i$ (equation 27). $f(x)$ is $-\sqrt{r^2-(x-h)^2}+k$ and $f'(x)$ is given by $\frac{x-h}{\sqrt{r^2-(x-h)^2}}$. Thus:

$$2A_i = \frac{2}{r^2} \tag{58}$$

$$-D_i = \frac{2x}{r^2} - \frac{\frac{2(x-h)}{\sqrt{r^2-(x-h)^2}}}{r\sqrt{1+\frac{(x-h)^2}{r^2-(x-h)^2}}} \tag{59}$$

Simplifying the equation 59

$$-D_i = \frac{2x}{r^2} - \frac{\frac{2(x-h)}{\sqrt{r^2-(x-h)^2}}}{r\sqrt{\frac{r^2-(x-h)^2+(x-h)^2}{r^2-(x-h)^2}}} \tag{60}$$

$$-D_i = \frac{2x}{r^2} - \frac{\frac{2(x-h)}{\sqrt{r^2-(x-h)^2}}}{\frac{r^2}{\sqrt{r^2-(x-h)^2}}} \tag{61}$$

$$-D_i = \frac{2h}{r^2} \tag{62}$$

The expressions for $-D_i$ (equation 62), $D_i^2 - 4A_iF_i$ (equation 57) and $2A_i$ (equation 58) are replaced in the equations 9 and 10:

$$x_{i+} = \frac{\frac{2h}{r^2} + \sqrt{\frac{4}{r^2} - \frac{4k^2}{r^4}}}{\frac{2}{r^2}} \tag{63}$$

$$x_{i+} = h + \sqrt{r^2 - k^2} \tag{64}$$

$$x_{i-} = \frac{\frac{2h}{r^2} - \sqrt{\frac{4}{r^2} - \frac{4k^2}{r^4}}}{\frac{2}{r^2}} \tag{65}$$

$$x_{i-} = h - \sqrt{r^2 - k^2} \tag{66}$$

The roots are imaginary if $r^2 - k^2$ is less than 0:

$$r^2 - k^2 < 0 \tag{67}$$

Simplifying equation 67:

$$r^2 < k^2 \tag{68}$$

$$r < |k| \tag{69}$$

$$k < -r \quad \text{and} \quad k > r \tag{70}$$

If the center at (h, k) is on the x-axis ($k = 0$) in the equations 64 and 66:

$$x_{i+} = h + r \tag{71}$$

$$x_{i-} = h - r \tag{72}$$

The previous analysis can also be done with second tangent circles of radius r to the functions $f(x) = \sqrt{r^2 - (x-h)^2} + k$ and $f(x) = -\sqrt{r^2 - (x-h)^2} + k$ (Figure 4). Similarly, the second tangent circles to the function $f(x) = \sqrt{r^2 - (x-h)^2} + k$ coincide with the original circle. The second tangent circles to the function $f(x) = -\sqrt{r^2 - (x-h)^2} + k$ generate an inverted crown-like graph, there some circles cross the line $y = k$ and others do not.

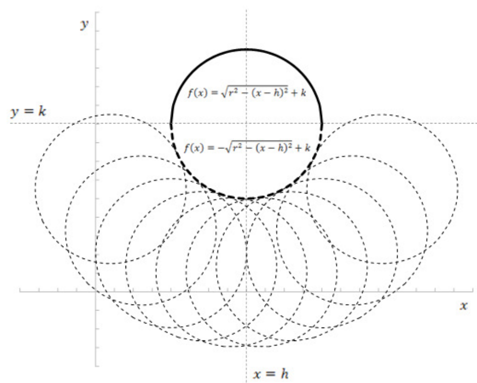


Figure 4: Several second tangent circles of radius r to the functions $f(x) = \sqrt{r^2 - (x-h)^2} + k$ and $f(x) = -\sqrt{r^2 - (x-h)^2} + k$.
Source: Own creation

The equation for the second tangent circle to the graph of a function $y = f(x)$ at a point $(x_i, f(x_i))$ can be written using the general second-degree equation (Gómez-Villarraga, 2021). The roots are calculated considering $y = 0$ obtaining the equation 5. The quadratic equation has two solutions (equations 9 and 10). The coefficients for the second tangent circle are given by:

$$A_i = \frac{1}{r^2} \tag{73}$$

$$D_i = -\frac{2f'(x_i)}{r\sqrt{1+[f'(x_i)]^2}} - \frac{2x_i}{r^2} \tag{74}$$

$$F_i = \frac{x_i^2 + [f(x_i)]^2}{r^2} - \frac{2[f'(x_i)x_i - f(x_i)]}{r\sqrt{1+[f'(x_i)]^2}} \tag{75}$$

(Gómez-Villarraga, 2021, p. 37)

The discriminant $D_i^2 - 4A_iF_i$ of the quadratic equation can be calculated using the equations 73-75

$$D_i^2 - 4A_iF_i = \frac{4[f'(x_i)]^2}{r^2\{1+[f'(x_i)]^2\}} + \frac{8f(x_i)}{r^3\sqrt{1+[f'(x_i)]^2}} - \frac{4[f(x_i)]^2}{r^4} \tag{76}$$

The second tangent circles to the function $f(x) = -\sqrt{r^2-(x-h)^2} + k$ are considered. $f(x)$ is $-\sqrt{r^2-(x-h)^2}+k$ and $f'(x)$ is given by $\frac{x-h}{\sqrt{r^2-(x-h)^2}}$. $x_i, f(x_i), f'(x_i)$ can be replaced by $x, f(x), f'(x)$ respectively in the equations 73-76 to obtain $D_i^2 - 4A_iF_i$, $2A_i$ and $-D_i$. Then, these results are replaced in the equations 9 and 10 for x_{i+} and x_{i-} and if the center at (h,k) is on the x-axis ($k = 0$), the same expressions as in the equations 36 and 37 are obtained.

The complex function given by the equation 50 with the domain given by the equation 51 is a circle in the complex plane with radius $r\sqrt{3}$ *(equation 49). The complex function represents the complex roots (discriminant less than zero) and the unique real solution (discriminant is equal to zero) of the second tangent circles of radius r to the real-valued function* $f(x) = -\sqrt{r^2-(x-h)^2}$ *that represents the inferior part of a circle with center on the x-axis at (h,0) and radius r.*

The Figure 5 shows the complex mapping of $h - \frac{\sqrt{3}}{2}r \leq Re(z) \leq h + \frac{\sqrt{3}}{2}r$ and $Im(z) = 0$ using the function $f(z) = 2z - h \pm \sqrt{4(z-h)^2 - 3r^2}$.

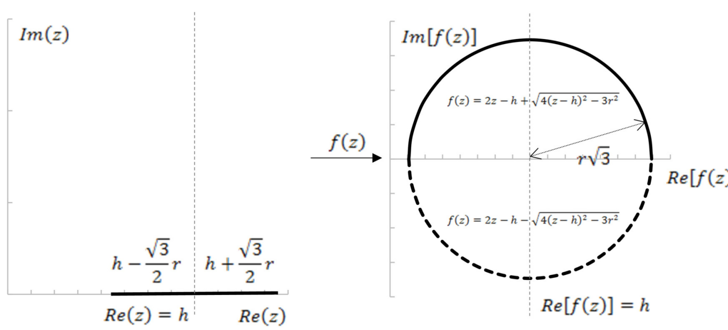


Figure 5: Complex mapping of $h - \frac{\sqrt{3}}{2}r \leq Re(z) \leq h + \frac{\sqrt{3}}{2}r$ and $Im(z) = 0$ using the function $f(z) = 2z - h \pm \sqrt{4(z-h)^2 - 3r^2}$.

Source: Own creation

3 Conclusions

The complex function $f(z) = 2z - h \pm \sqrt{4(z-h)^2 - 3r^2}$ with the domain $h - \frac{\sqrt{3}}{2}r \leq Re(z) \leq h + \frac{\sqrt{3}}{2}r$ and $Im(z) = 0$ is a circle in the complex plane with radius $r\sqrt{3}$. The complex function represents the complex roots (discriminant less than zero) and the unique real solution (discriminant is equal to zero) of the first (or second) tangent circles of radius r to the real-valued function $f(x) = \sqrt{r^2 - (x-h)^2}$ (or $f(x) = -\sqrt{r^2 - (x-h)^2}$) that represents the superior (or inferior) part of a circle with center on the x-axis at $(h, 0)$ and radius r.

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